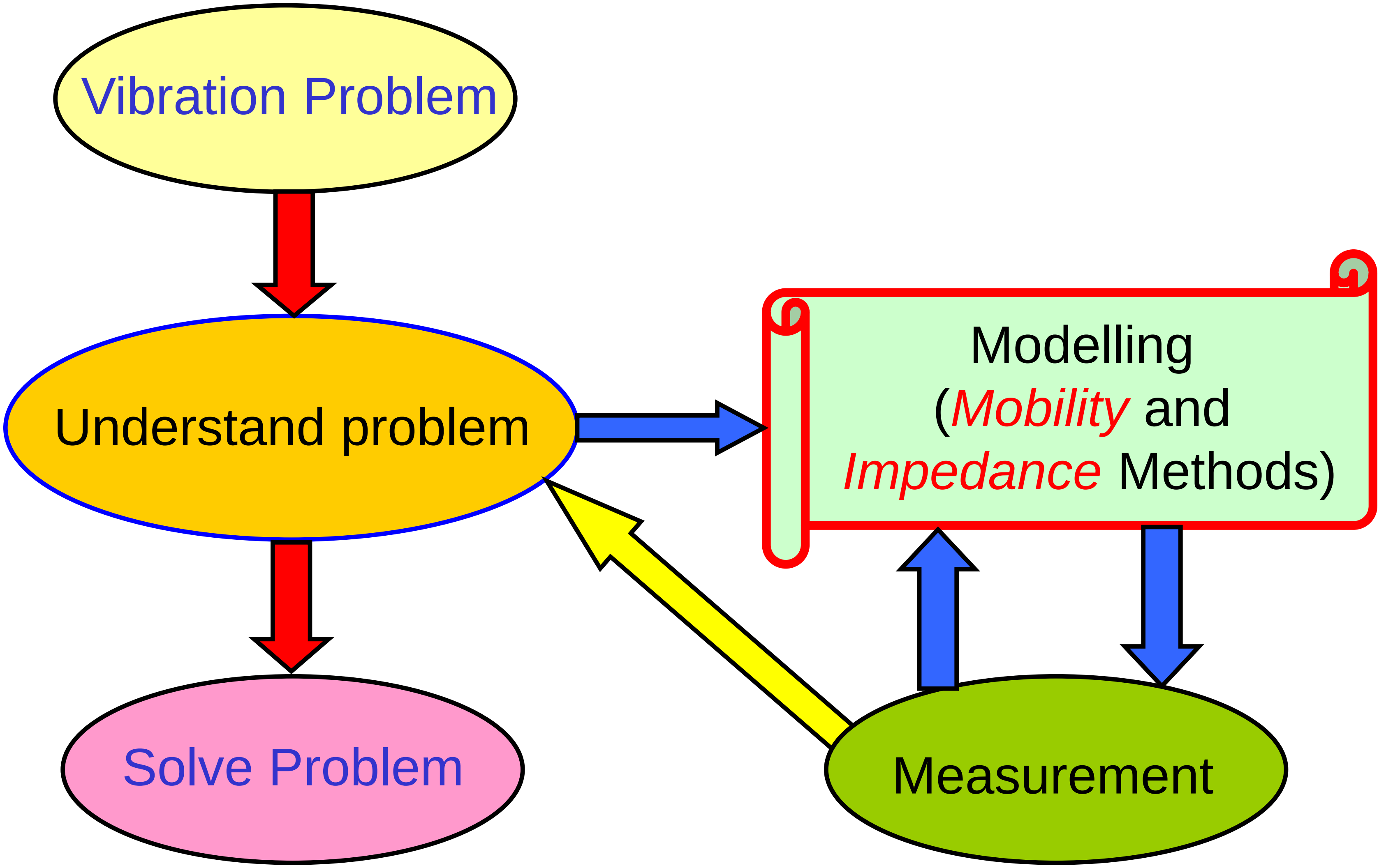
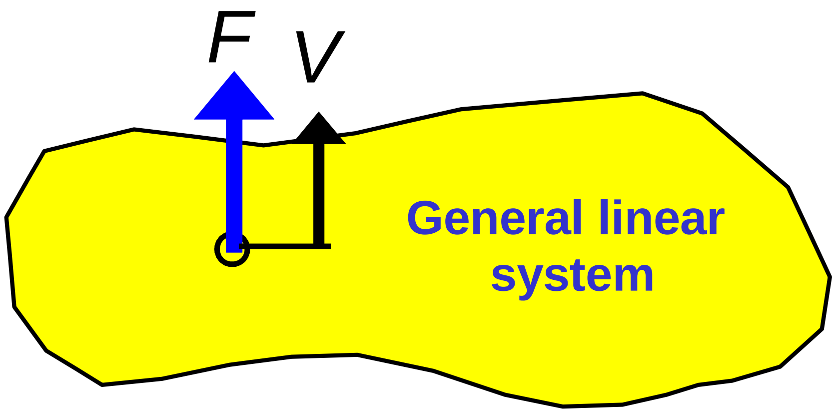
Mobility and Impedance Methods

Vibration control



Mobility and Impedance

- The response of a structure to a harmonic force can be expressed in terms of its mobility or impedance



At frequency ω the velocity can be written in complex notation $v(t) = Ve^{j\omega t}$

V is the complex amplitude.

Similarly for the force $f(t) = Fe^{j\omega t}$

The ***mobility*** is defined as

$$\text{Mobility} = \frac{V(j\omega)}{F(j\omega)}$$

The ***impedance*** is defined as

$$\text{Impedance} = \frac{F(j\omega)}{V(j\omega)}$$

- If the force and velocity are at the same point this is a ‘point’ mobility
- If they are at different points it is a ‘transfer’ mobility

Note that both mobility and impedance are frequency domain quantities

Frequency Response Functions (FRFs)

$$\text{Accelerance} = \frac{\text{Acceleration}}{\text{Force}}$$

$$\text{Apparent Mass} = \frac{\text{Force}}{\text{Acceleration}}$$

$$\text{Mobility} = \frac{\text{Velocity}}{\text{Force}}$$

$$\text{Impedance} = \frac{\text{Force}}{\text{Velocity}}$$

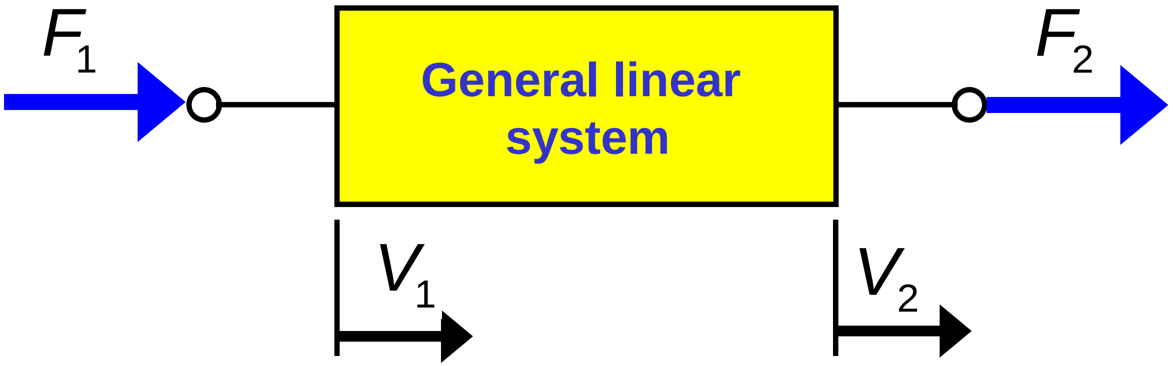
$$\text{Receptance} = \frac{\text{Displacement}}{\text{Force}}$$

$$\text{Dynamic Stiffness} = \frac{\text{Force}}{\text{Displacement}}$$

Mobility and Impedance Methods

- The total response of a set of coupled components can be expressed in terms of the mobility of the individual components

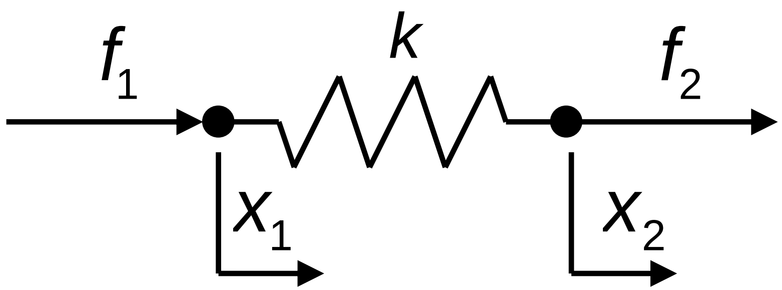
- In the simplest case each component has two inputs (one at each end) which permit coupling



- The two parameters at each input point are force, *F*, and velocity, *V*.

Simple Idealised Elements

- **Spring**



$$f_1 = k\left(x_1 - x_2\right)$$

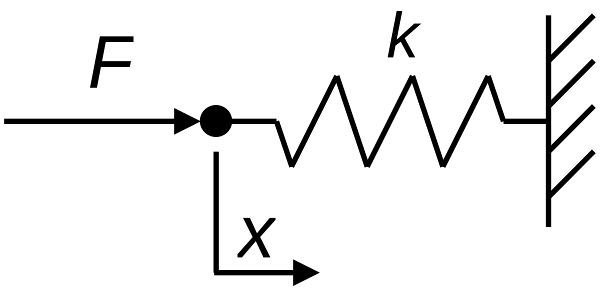
$$f_2 = k\left(x_2 - x_1\right)$$

$$f_1 = -f_2$$

- no mass
- force passes through it unattenuated

Assume $f = Fe^{j\omega t}$ and $x = Xe^{j\omega t}$

Also, block one end so that $x_2 = 0$



So $F = kX$

Because $X = \dfrac{V}{j\omega}$ then $F = \dfrac{KV}{j\omega}$

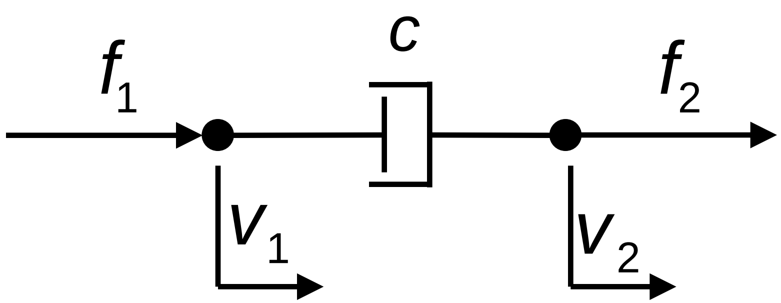
So the impedance of a spring is given by

$$\boxed{Z_k = \frac{F}{V} = \frac{k}{j\omega}}$$

Note that the force is in quadrature with the velocity. Thus a spring is a ***reactive*** element that ***does not dissipate energy***

Simple Idealised Elements

- **Viscous damper**

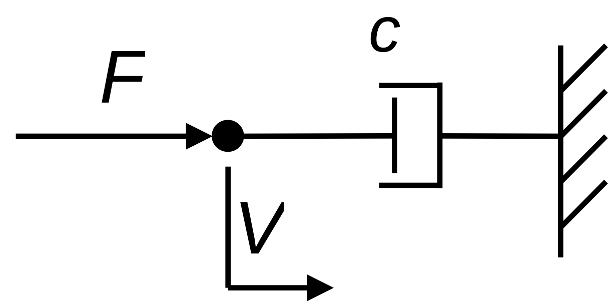


$$f_1 = c\left(v_1 - v_2\right)$$
$$f_2 = c\left(v_2 - v_1\right)$$
$$f_1 = -f_2$$

- no mass or elasticity
- force passes through it unattenuated

Assume $f = Fe^{j\omega t}$ and $v = Ve^{j\omega t}$

Also, block one end so that $v_2 = 0$



So $F = cV$

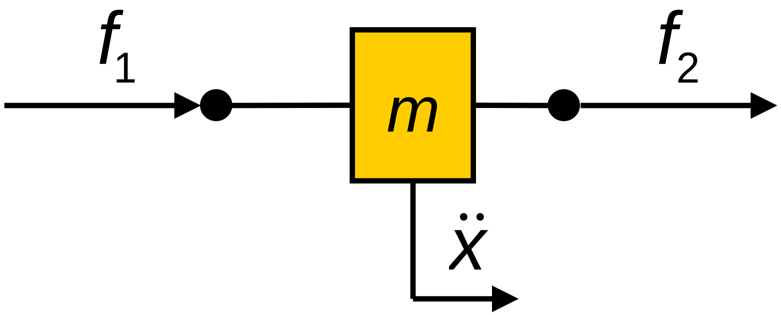
So the impedance of a damper is given by

$$\boxed{Z_c = \frac{F}{V} = c}$$

Note that the force is in phase with the velocity. Thus a damper is a ***resistive*** element that ***dissipates energy***

Simple Idealised Elements

- **Mass**



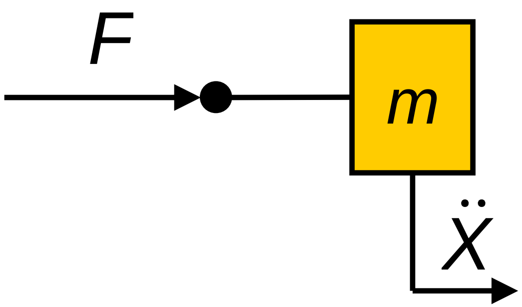
$$f_1 + f_2 = m\ddot{x}$$

$$f_2 = m\ddot{x} - f_1$$

- rigid
- force ***does not*** pass through it unattenuated

Assume $f = Fe^{j\omega t}$ and $\ddot{x} = \ddot{X}e^{j\omega t}$

Also, set one end to be free so that $f_2 = 0$



So $F = m\ddot{X}$

Because $\ddot{X} = j\omega V$ then $F = j\omega mV$

So the impedance of a mass is given by

$$\boxed{Z_m = \frac{F}{V} = j\omega m}$$

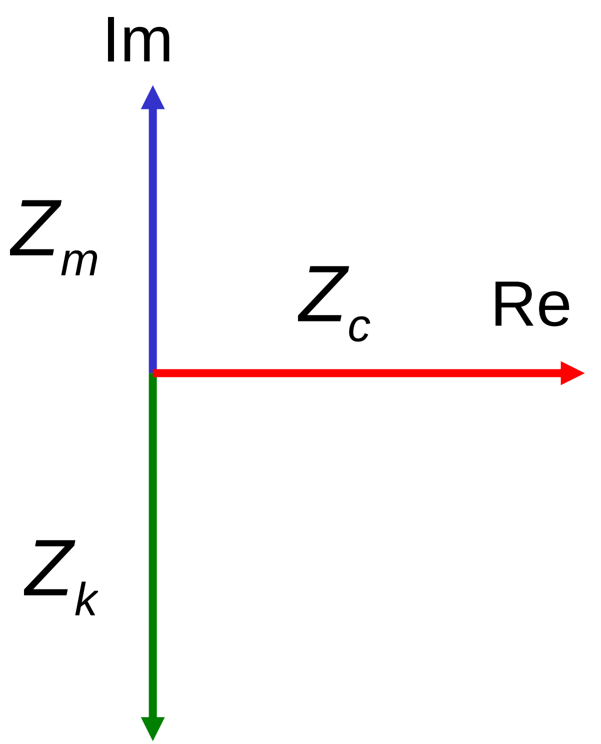
Note that the force is in quadrature with the velocity. Thus a mass is a ***reactive*** element that ***does not dissipate energy***

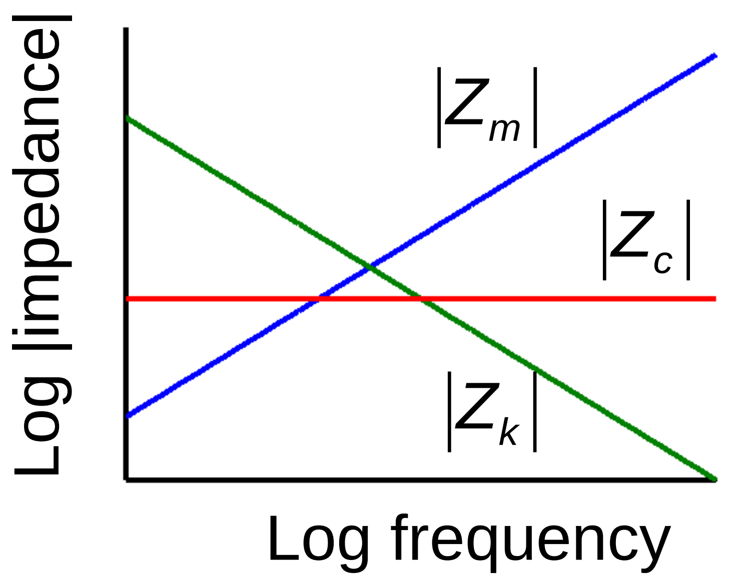
Impedances of Simple Elements - Summary

- **Spring** $Z_k = \frac{k}{j\omega} = \frac{-jk}{\omega}$

- **Damper** $Z_c = c$

- **Mass** $Z_m = j\omega m$



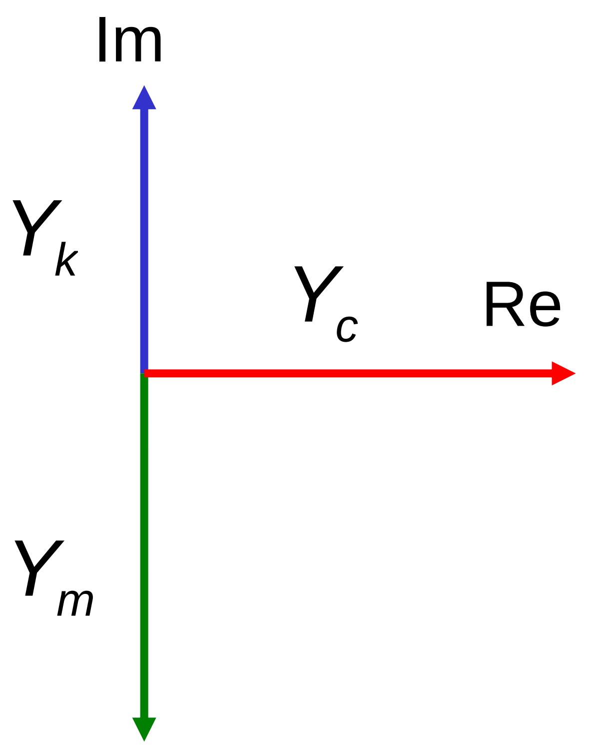


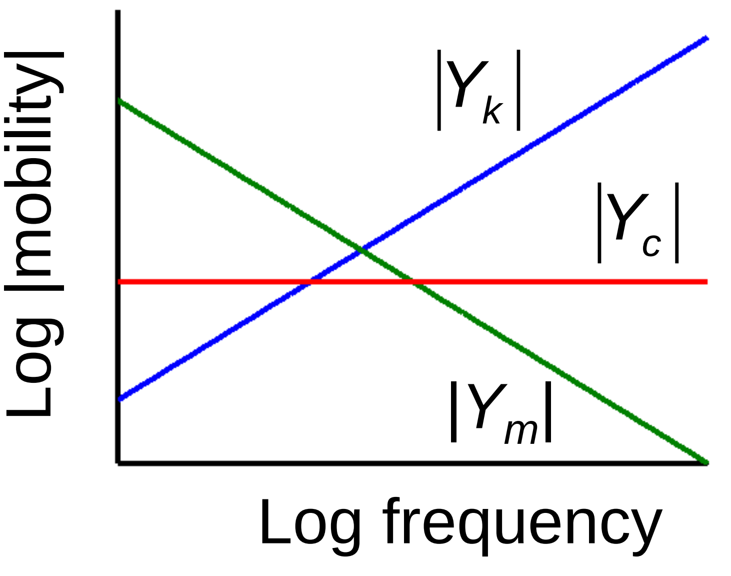
Mobilities of Simple Elements - Summary

- **Spring** $Y_k = \dfrac{j\omega}{k}$

- **Damper** $Y_c = \dfrac{1}{c}$

- **Mass** $Y_m = \dfrac{1}{j\omega m} = \dfrac{-j}{\omega m}$





Examples of impedance / mobility

mass	$F = m\ddot{X}$	$Z_{\text{mass}} = j\omega m$	$Y_{\text{mass}} = \frac{-j}{\omega m}$
spring	$F = kX$	$Z_{\text{spring}} = \frac{-jk}{\omega}$	$Y_{\text{spring}} = \frac{j\omega}{k}$
damper	$F = c\dot{X}$	$Z_{\text{damper}} = c$	$Y_{\text{damper}} = \frac{1}{c}$
infinite beam		$Z_{\text{beam}} = 2(1+j)\,\omega^{1/2}(EI)^{1/4}(\rho A)^{3/4}$	
infinite plate		$Z_{\text{plate}} = 8h^2\sqrt{\frac{E\rho}{12(1-\nu^2)}} = 2.3 c_L \rho h^2$	

Area A, second moment of area I, thickness h, Young's modulus E, density ρ, Poisson's ratio ν

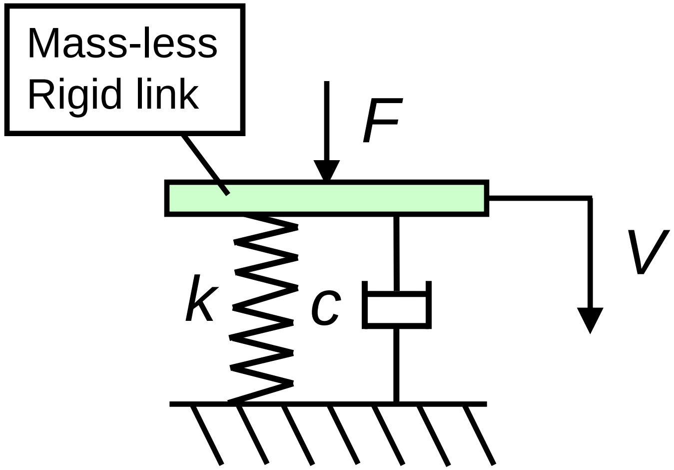
$c_L = \sqrt{E/\rho(1-\nu^2)}$

Notes on impedance / mobility

- Real part of *point* impedance (or mobility) is always **positive** (dissipation). Imaginary part can be positive (mass-like) or negative (spring-like).
- Infinite plate impedance is real and independent of frequency (equivalent to a damper).
- Beam impedance has a damper part and a mass part, both frequency dependent.
- Impedance/mobility of a finite structure tends to that of the equivalent infinite structure at high frequency and/or high damping.

Connecting Simple Elements

- **Adding Elements in *Parallel***



- **Same velocity**

$$F_1 = Z_1 V \qquad F_2 = Z_2 V$$

- **Shared force**

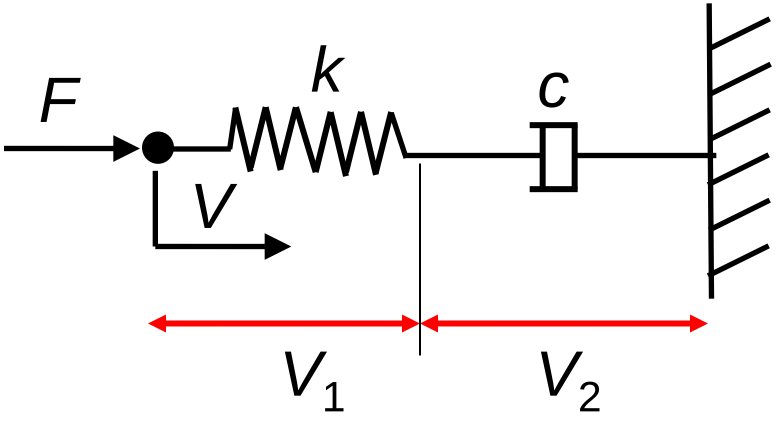
$$F = F_1 + F_2$$

$$F = (Z_1 + Z_2)V$$

$$\boxed{Z_{\text{total}} = \sum_{j=1}^{N} Z_j}$$

Connecting Simple Elements

- Adding Elements in ***Series***



- Same force

$$V_1 = Y_1 F \qquad V_2 = Y_2 F$$

- Shared velocity

$$V = V_1 + V_2$$

$$V = (Y_1 + Y_2)F$$

$$\boxed{Y_{\text{total}} = \sum_{j=1}^{N} Y_j}$$

Connecting Simple Elements

- **Adding Elements in *Parallel***

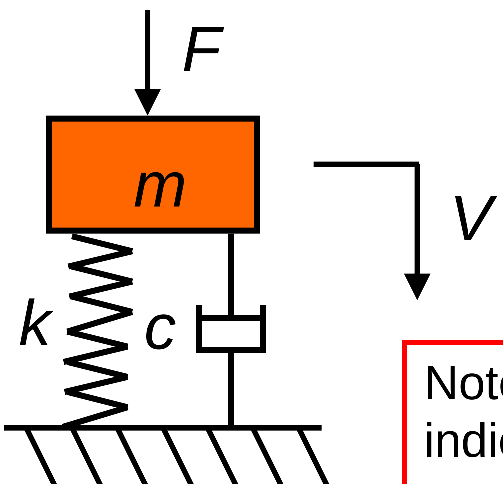
- **Impedances**

$$Z_{\text{total}} = \sum_{j=1}^{N} Z_j$$

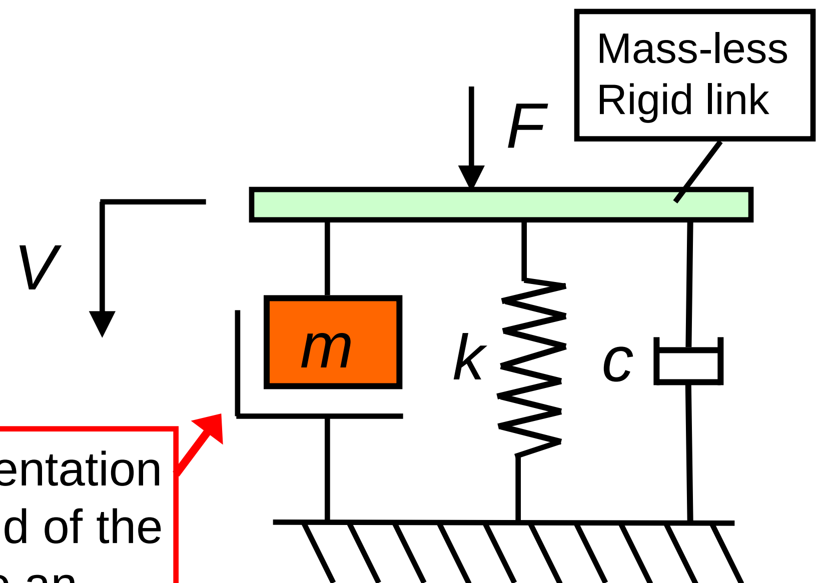
- **Mobilities**

$$\frac{1}{Y_{\text{total}}} = \sum_{j=1}^{N} \frac{1}{Y_j}$$

- **Example - SDOF System**



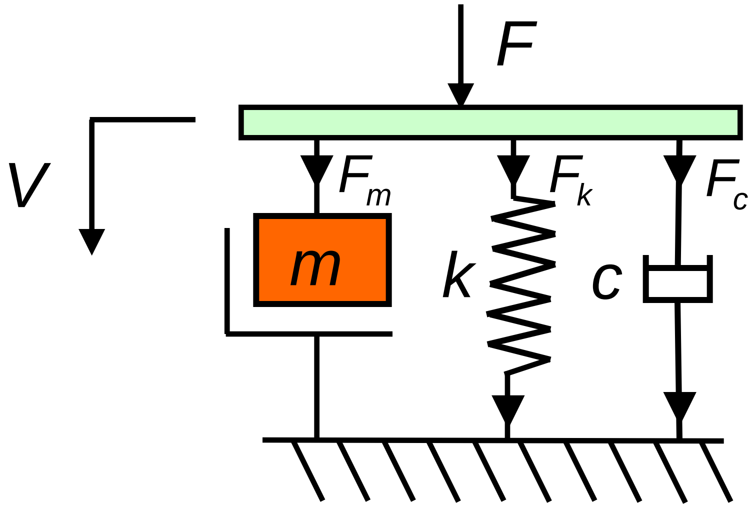
$\equiv$



Note that this representation indicates that one end of the mass is connected to an inertial reference point (F=0)

Connecting Simple Elements

- Adding Elements in ***Parallel***



$$F = F_m + F_k + F_c$$

- **Point Impedance**

$$Z_{11} = j\omega m + \frac{k}{j\omega} + c$$

- **Point Mobility**

$$Y_{11} = \frac{1}{j\omega m + \dfrac{k}{j\omega} + c}$$

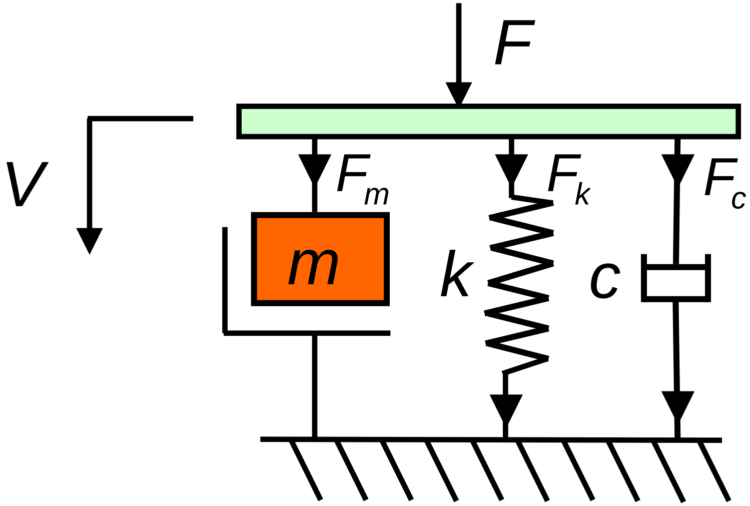
$$= \frac{j\omega}{k - \omega^2 m + j\omega c}$$

At low frequency stiffness dominates

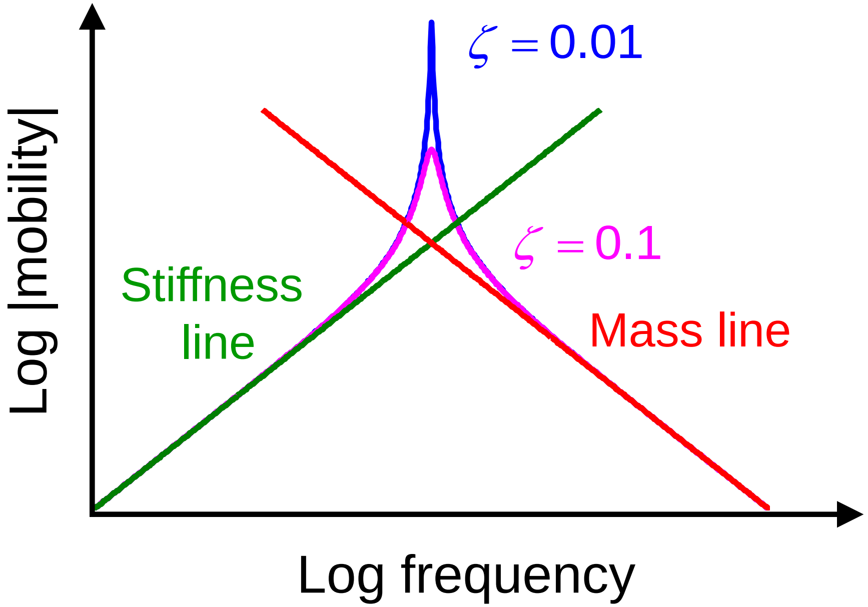
At resonance damping dominates

At high frequency mass dominates

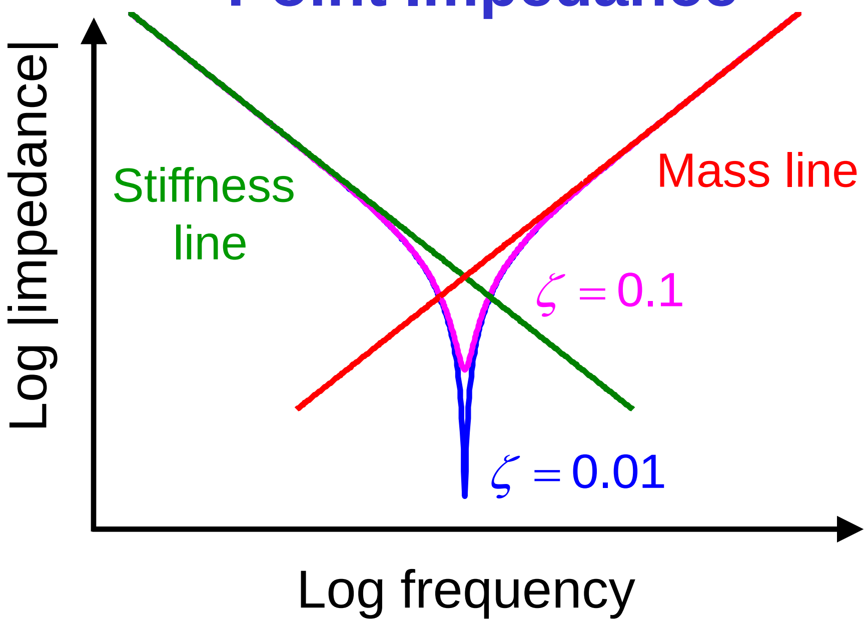
Frequency Response Functions



- **Point Mobility**



- **Point Impedance**



Connecting Simple Elements

- **Adding Elements in *Series***

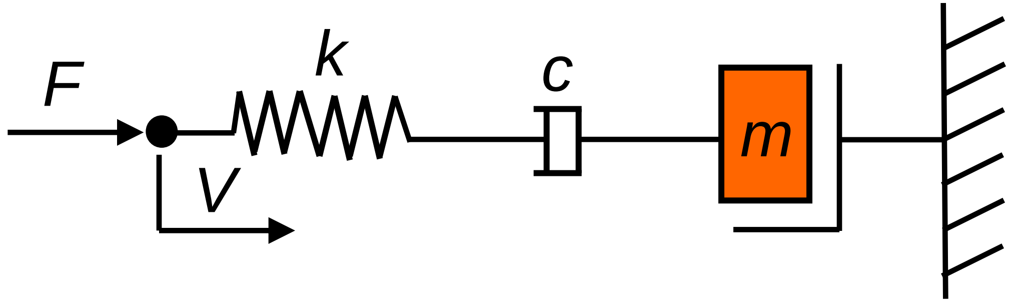
- **Impedances**

$$\frac{1}{Z_{\text{total}}} = \sum_{j=1}^{N} \frac{1}{Z_j}$$

- **Mobilities**

$$Y_{\text{total}} = \sum_{j=1}^{N} Y_j$$

- **Example**



- **Point Mobility**

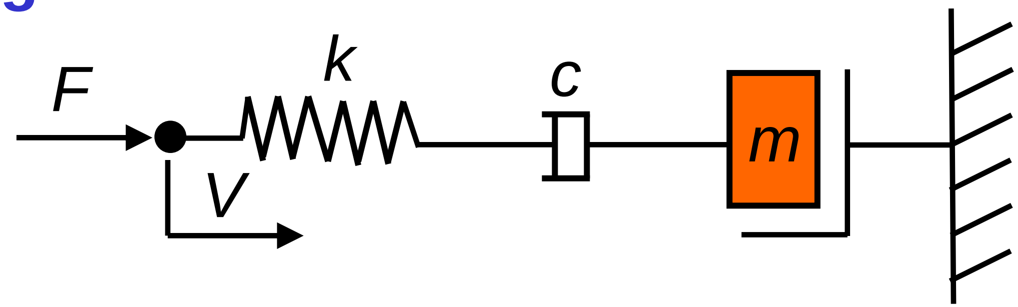
$$Y_{11} = \frac{j\omega}{k} + \frac{1}{c} + \frac{1}{j\omega m}$$

- **Point Impedance**

$$Z_{11} = \frac{1}{\frac{j\omega}{k} + \frac{1}{c} + \frac{1}{j\omega m}}$$

Connecting Simple Elements

- **Adding Elements in *Series***

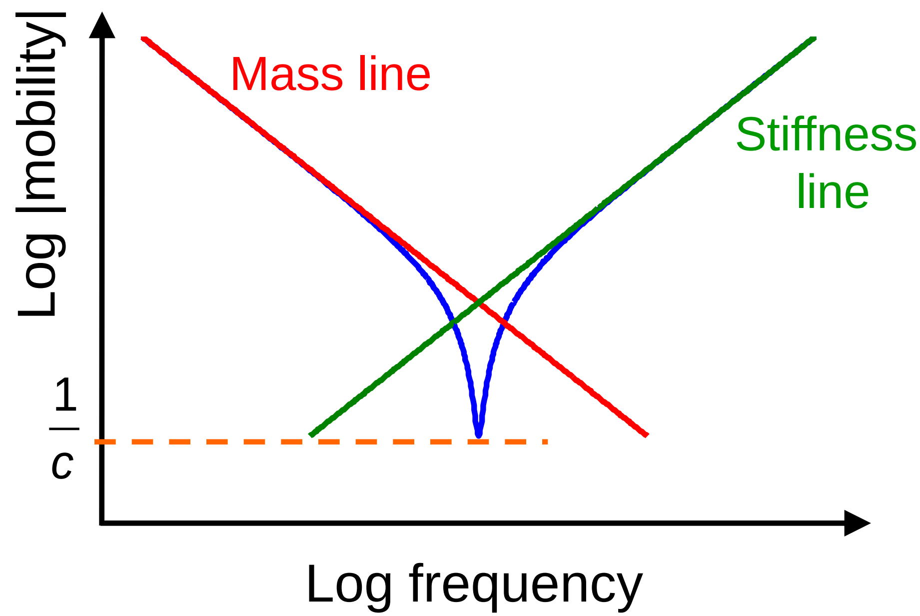


- **Point mobility**

At low frequency mass dominates

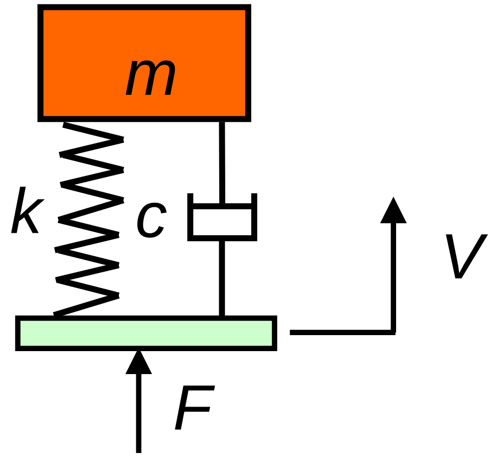
At resonance damping dominates

At high frequency stiffness dominates

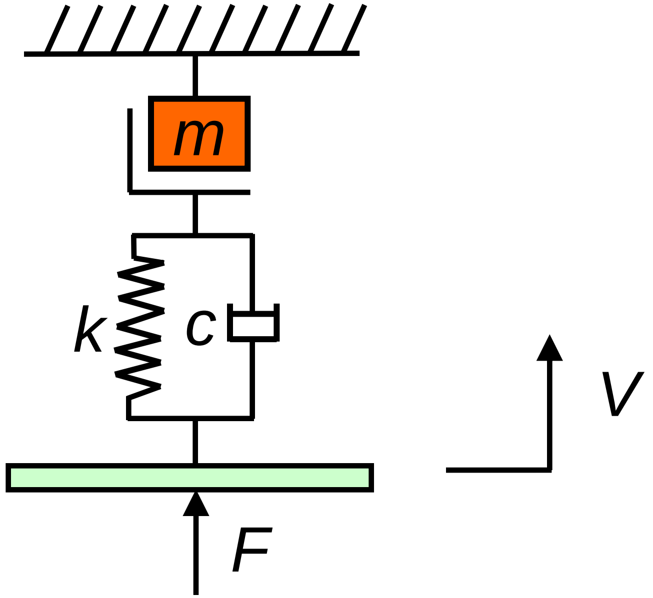


Adding a Combination of *Parallel* and *Series* Elements

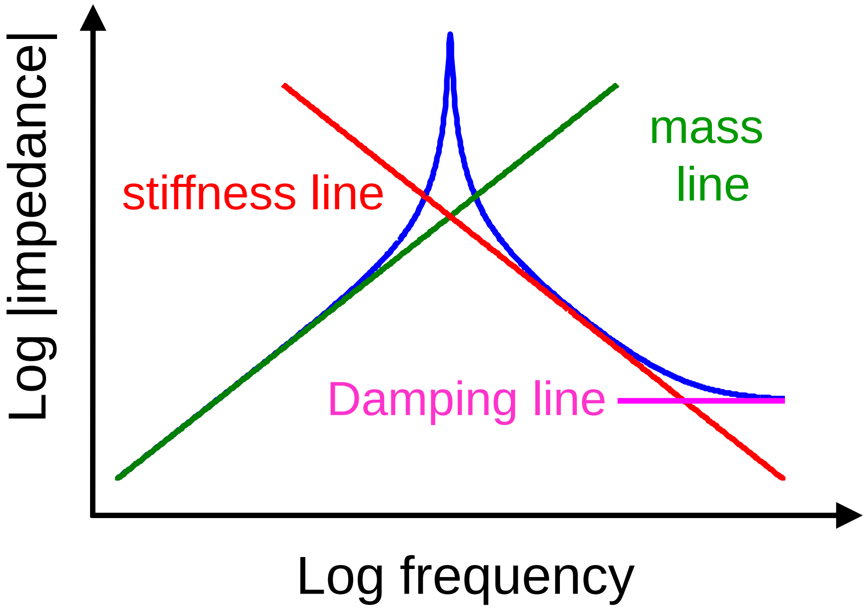
- **Example - SDOF System**



$\equiv$



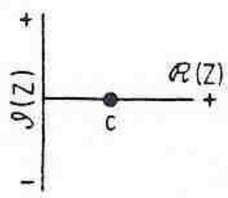
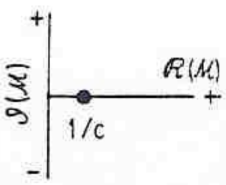
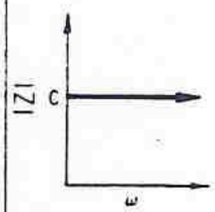
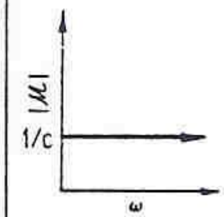
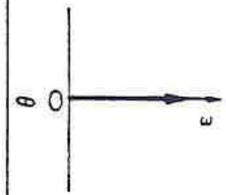
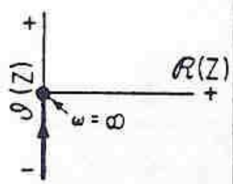
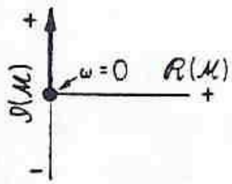
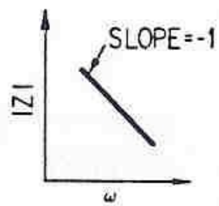
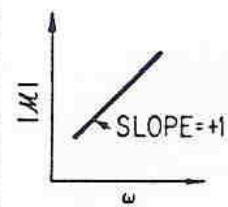
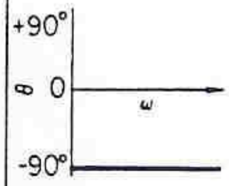
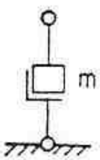
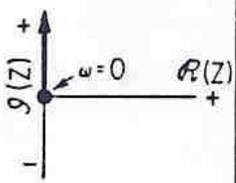
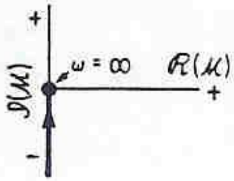
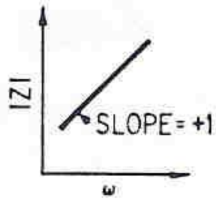
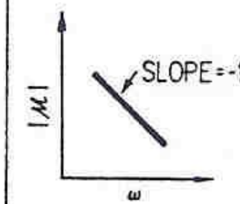
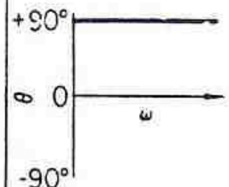
$$Z_{\text{total}} = \frac{1}{\left(\frac{1}{Z_m}\right) + \left(\frac{1}{Z_k + Z_c}\right)}$$



E.L.Hixson, Chapter 10 in Shock and Vibration Handbook

Table 10.2. Driving-point Impedance and Mobility of Ideal Mechanical Elements and Lumped Parameter Systems

When the system of elements shown includes a mass, the impedance or mobility given is the relationship between force and velocity at the one available connection, the other connection being attached to the inertial reference plane. When no mass is included, the impedance or mobility describes the relation between the force applied to the two system connections and the resulting relative velocity between these connections. Graphs of magnitude of impedance vs. ω and magnitude of mobility vs. ω are plotted on a log-log scale.

DIAGRAM OF SYSTEM	MATHEMATIC FORMULAS: IMPEDANCE Z - EQ.(10.7) MOBILITY $\mathcal{M}$ - EQ.(10.9)	IMPEDANCE IN THE COMPLEX PLANE	MOBILITY IN THE COMPLEX PLANE	MAGNITUDE OF IMPEDANCE	MAGNITUDE OF MOBILITY	IMPEDANCE ANGLE θ FIG.10.34
1. 	$Z = c$ $\mathcal{M} = 1/c$					
2. 	$Z = \frac{k}{j\omega}$ $\mathcal{M} = \frac{j\omega}{k}$					
3. 	$Z = j\omega m$ $\mathcal{M} = \frac{1}{j\omega m}$					

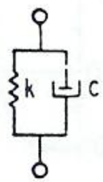
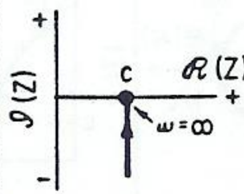
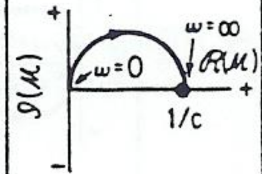
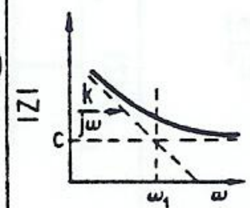
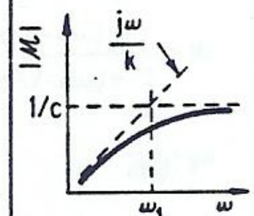
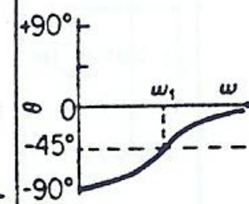
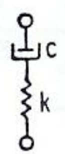
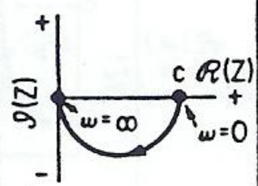
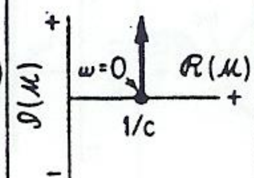
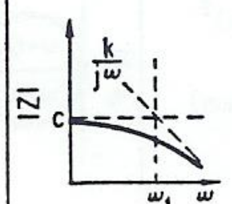
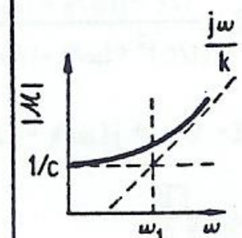
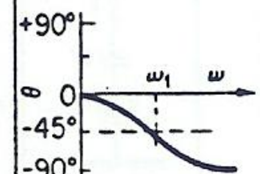
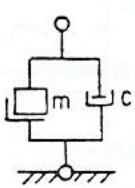
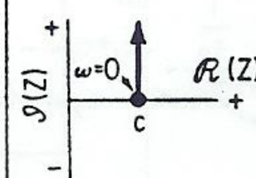
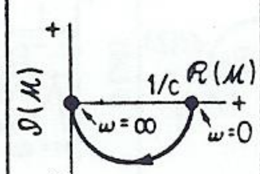
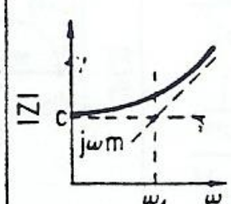
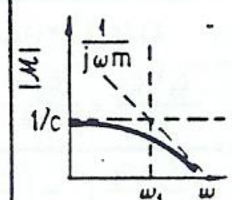
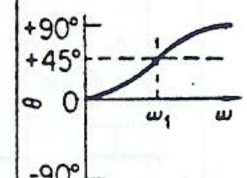
<table>
<tr><th>DIAGRAM OF SYSTEM</th><th>MATHEMATIC FORMULAS:
IMPEDANCE Z - EQ. (10.7)
MOBILITY $\mathcal{M}$ - EQ. (10.9)</th><th>IMPEDANCE IN THE COMPLEX PLANE</th><th>MOBILITY IN THE COMPLEX PLANE</th><th>MAGNITUDE OF IMPEDANCE</th><th>MAGNITUDE OF MOBILITY</th><th>IMPEDANCE ANGLE θ
FIG. 10.34</th></tr>
<tr><td>4.
</td><td>$Z = c + \frac{k}{j\omega}$
$\mathcal{M} = \frac{c - \frac{k}{j\omega}}{c^2 + (k/\omega)^2}$
$\omega_1 = \frac{k}{c}$</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>5.
</td><td>$Z = \frac{1/c - j\omega/k}{(1/c)^2 + (\omega/k)^2}$
$\mathcal{M} = \frac{1}{c} + \frac{j\omega}{k}$
$\omega_1 = \frac{k}{c}$</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>6.
</td><td>$Z = c + j\omega m$
$\mathcal{M} = \frac{c - j\omega m}{c^2 + \omega^2 m^2}$
$\omega_1 = \frac{c}{m}$</td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

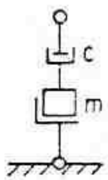
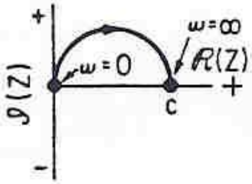
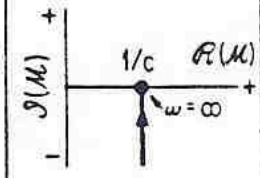
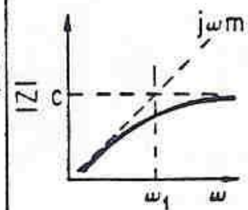
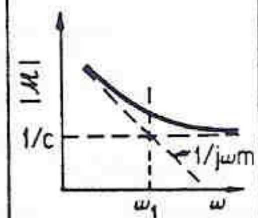
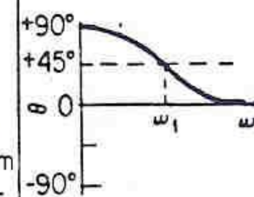
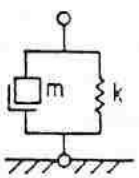
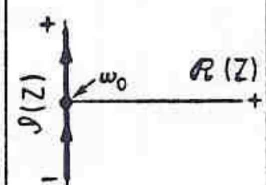
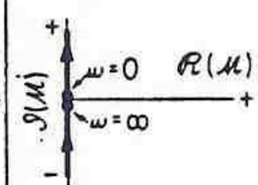
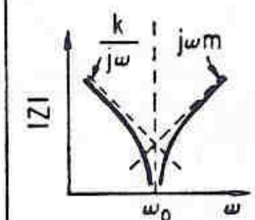
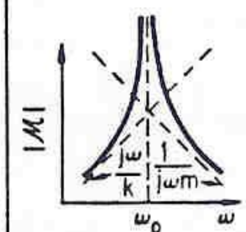
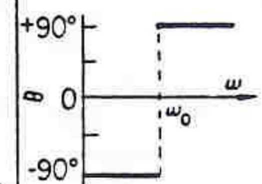
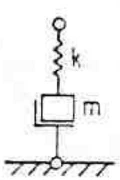
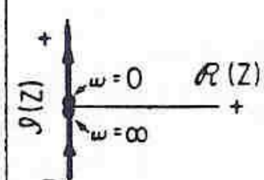
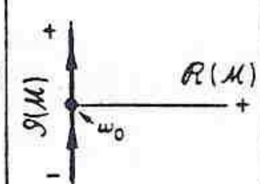
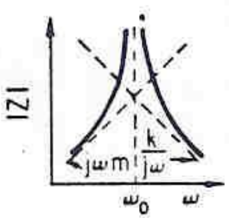
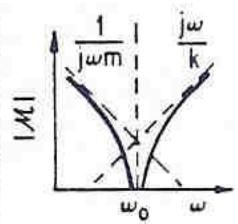
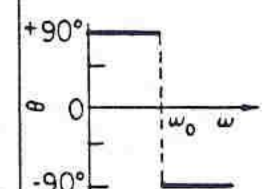
DIAGRAM OF SYSTEM	MATHEMATIC FORMULAS: IMPEDANCE Z - EQ.(10.7) MOBILITY $\mathcal{M}$ - EQ.(10.9)	IMPEDANCE IN THE COMPLEX PLANE	MOBILITY IN THE COMPLEX PLANE	MAGNITUDE OF IMPEDANCE	MAGNITUDE OF MOBILITY	IMPEDANCE ANGLE θ FIG.10.34
7. 	$Z = \frac{1/c + j/\omega m}{(1/c)^2 + (1/\omega m)^2}$ $\mathcal{M} = 1/c + 1/j\omega m$ $\omega_1 = \frac{c}{m}$					
8. 	$Z = j\omega m + \frac{k}{j\omega}$ $\mathcal{M} = \frac{-j}{\omega m - k/\omega}$ $\omega_0 = \sqrt{\frac{k}{m}}$					
9. 	$Z = \frac{-j}{\omega/k - 1/\omega m}$ $\mathcal{M} = j\omega/k + 1/j\omega m$ $\omega_0 = \sqrt{\frac{k}{m}}$					

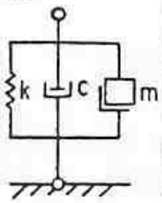
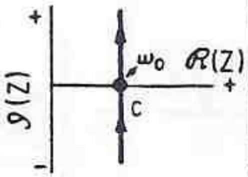
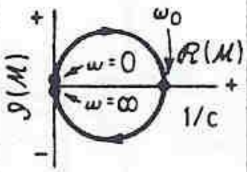
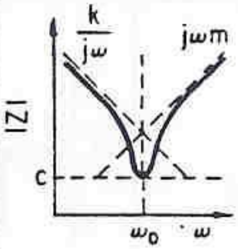
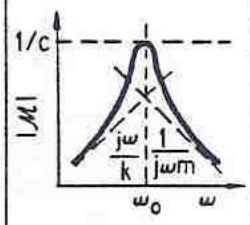
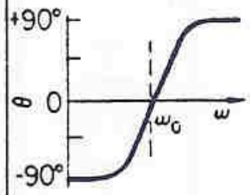
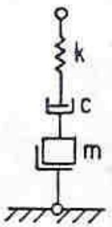
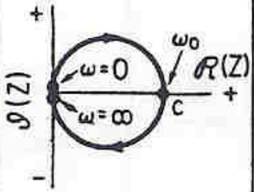
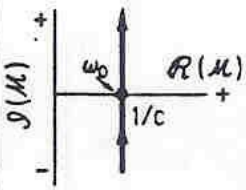
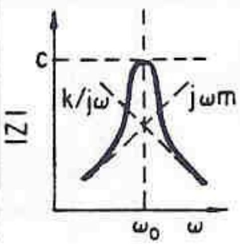
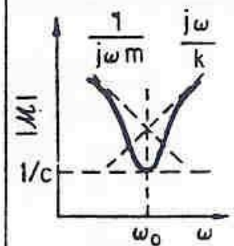
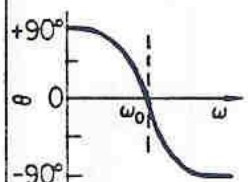
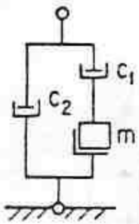
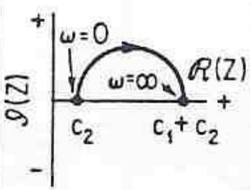
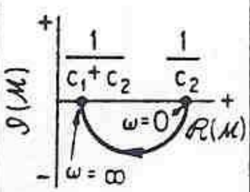
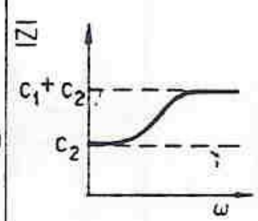
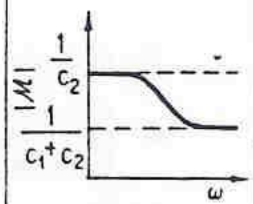
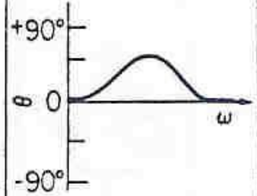
DIAGRAM OF SYSTEM	MATHEMATIC FORMULAS: IMPEDANCE Z - EQ.(10.7) MOBILITY $\mathcal{M}$ - EQ.(10.9)	IMPEDANCE IN THE COMPLEX PLANE	MOBILITY IN THE COMPLEX PLANE	MAGNITUDE OF IMPEDANCE	MAGNITUDE OF MOBILITY	IMPEDANCE ANGLE θ FIG. 10.34
10. 	$Z = c + j(\omega m - k/\omega)$ $\mathcal{M} = \dfrac{c - j(\omega m - k/m)}{c^2 + (\omega m - k/m)^2}$ $\omega_0 = \sqrt{\dfrac{k}{m}}$					
11. 	$Z = \dfrac{1/c - j(\omega/k - 1/\omega m)}{(1/c)^2 + (\omega/k - 1/\omega m)^2}$ $\mathcal{M} = 1/c + j(\omega/k - 1/\omega m)$ $\omega_0 = \sqrt{\dfrac{k}{m}}$					
12. 	$Z = \dfrac{\dfrac{c_1 + c_2}{c_1^2} + \dfrac{c_2}{\omega^2 m^2} + \dfrac{j}{\omega m}}{(1/c_1)^2 + (1/\omega m)^2}$ $\mathcal{M} = \dfrac{\dfrac{c_1 + c_2}{c_1^2} + \dfrac{c_2}{\omega^2 m^2} - \dfrac{j}{\omega m}}{\left(\dfrac{c_1 + c_2}{c_1}\right)^2 + \left(\dfrac{c_2}{\omega m}\right)^2}$					

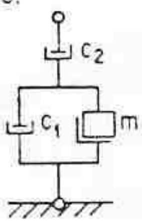
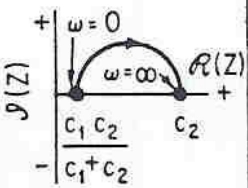
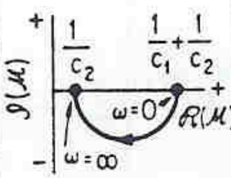
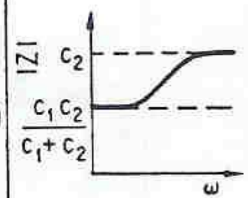
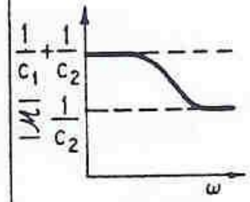
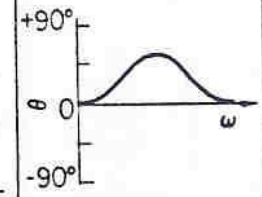
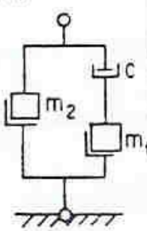
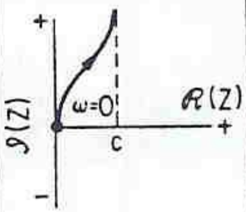
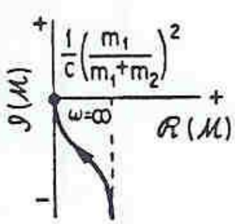
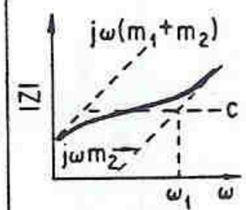
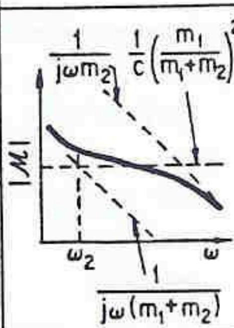
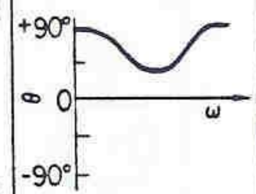
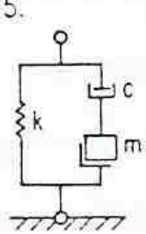
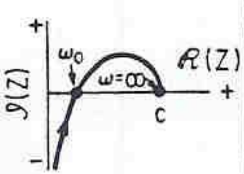
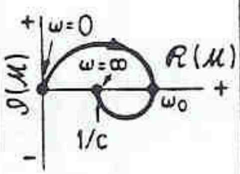
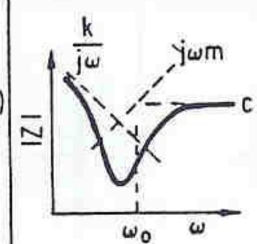
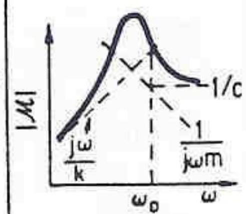
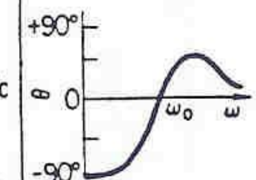
DIAGRAM OF SYSTEM	MATHEMATIC FORMULAS: IMPEDANCE Z - EQ. (10.7) MOBILITY $\mathcal{M}$ - EQ. (10.9)	IMPEDANCE IN THE COMPLEX PLANE	MOBILITY IN THE COMPLEX PLANE	MAGNITUDE OF IMPEDANCE	MAGNITUDE OF MOBILITY	IMPEDANCE ANGLE θ FIG. 10.34
13. 	$Z = \dfrac{c_1\left(\dfrac{c_1+c_2}{c_2}\right) + \dfrac{\omega^2 m^2}{c_2} + j\omega m}{\left(\dfrac{c_1+c_2}{c_2}\right)^2 + \left(\dfrac{\omega m}{c_2}\right)^2}$ $\mathcal{M} = \dfrac{c_1\left(\dfrac{c_1+c_2}{c_2}\right) + \dfrac{\omega^2 m^2}{c_2} - j\omega m}{c_1^2 + \omega^2 m^2}$					
14. 	$Z = \dfrac{\dfrac{1}{c} + j\left[\dfrac{\omega m_2}{c^2} + \dfrac{1}{\omega m_1}\left(\dfrac{m_1+m_2}{m_1}\right)\right]}{(1/c)^2 + (1/\omega m_1)^2}$ $\mathcal{M} = \dfrac{\dfrac{1}{c} - j\left[\dfrac{\omega m_2}{c^2} + \dfrac{1}{\omega m_1}\left(\dfrac{m_1+m_2}{m_1}\right)\right]}{\left(\dfrac{m_1+m_2}{m_1}\right)^2 + \left(\dfrac{\omega m_2}{c}\right)^2}$ $\omega_1 = \dfrac{c}{m_2}$ $\quad \omega_2 = \dfrac{c}{m_1}\left(\dfrac{m_1+m_2}{m_1}\right)$					
15. 	$Z = \dfrac{\dfrac{1}{c} + \dfrac{j}{\omega}\left[\dfrac{1}{m} - k\left(\dfrac{1}{c^2} + \dfrac{1}{\omega^2 m^2}\right)\right]}{(1/c)^2 + (1/\omega m)^2}$ $\mathcal{M} = \dfrac{\dfrac{\omega^2 m^2}{c} + j\left(\dfrac{\omega m^2 k}{c} + \dfrac{k}{\omega} - \omega m\right)}{(mk/c)^2 + (\omega m - k/\omega)^2}$ $\omega_0 = \sqrt{\dfrac{k}{m - \dfrac{km^2}{c^2}}}$					

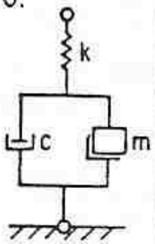
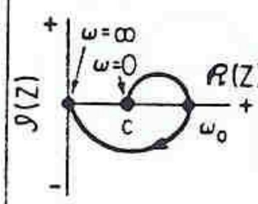
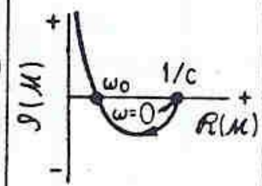
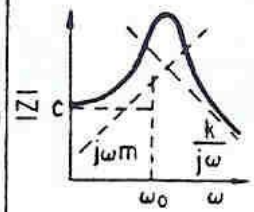
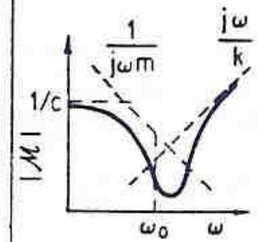
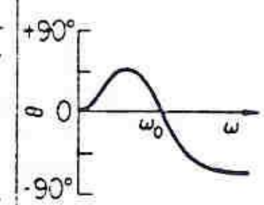
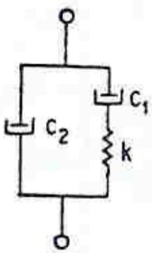
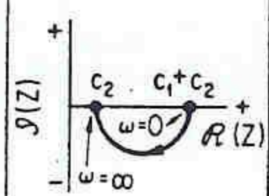
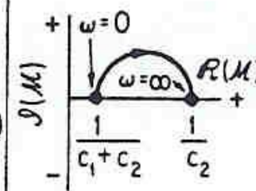
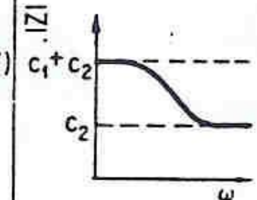
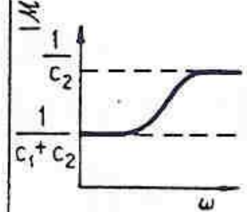
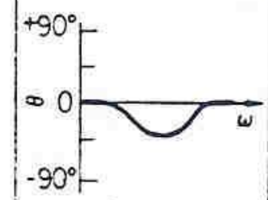
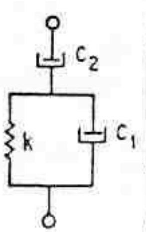
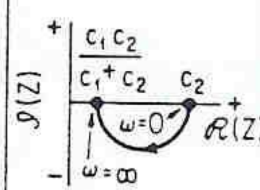
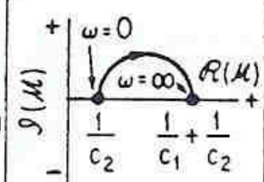
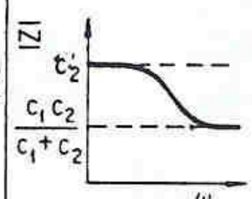
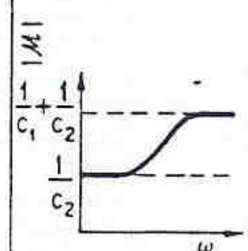
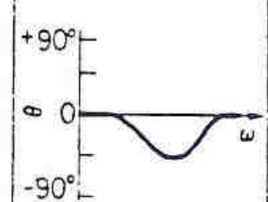
DIAGRAM OF SYSTEM	MATHEMATIC FORMULAS: IMPEDANCE Z - EQ. (10.7) MOBILITY $\mathcal{M}$ - EQ. (10.9)	IMPEDANCE IN THE COMPLEX PLANE	MOBILITY IN THE COMPLEX PLANE	MAGNITUDE OF IMPEDANCE	MAGNITUDE OF MOBILITY	IMPEDANCE ANGLE θ FIG. 10.34
16. 	$Z=\dfrac{\dfrac{ck^2}{\omega^2}-jkm\left[(\omega m-k/\omega)+\dfrac{c^2k}{\omega m}\right]}{c^2+(\omega m-k/\omega)^2}$ $\mathcal{M}=\dfrac{c+j\omega\left(\dfrac{c^2+\omega^2m^2}{k}-m\right)}{c^2+\omega^2m^2}$ $\omega_0=\sqrt{\dfrac{k}{m}-\dfrac{c^2}{m^2}}$					
17. 	$Z=\dfrac{\dfrac{c_1+c_2}{c_1^2}+\dfrac{c_2\omega^2}{k^2}-j\dfrac{\omega}{k}}{(1/c_1)^2+(\omega/k)^2}$ $\mathcal{M}=\dfrac{\dfrac{c_1+c_2}{c_1^2}+\dfrac{c_2\omega^2}{k^2}+\dfrac{j\omega}{k}}{\left(\dfrac{c_1+c_2}{c_1}\right)^2+\left(\dfrac{c_2\omega}{k}\right)^2}$					
18. 	$Z=\dfrac{c_1\left(\dfrac{c_1+c_2}{c_2}\right)+\dfrac{k^2}{c_2\omega^2}-j\dfrac{k}{\omega}}{\left(\dfrac{c_1+c_2}{c_2}\right)^2+\left(\dfrac{k}{\omega c_2}\right)^2}$ $\mathcal{M}=\dfrac{c_1\left(\dfrac{c_1+c_2}{c_2}\right)+\dfrac{k^2}{c_2\omega^2}+\dfrac{jk}{\omega}}{c_1^2+(k/\omega)^2}$					

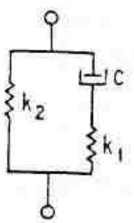
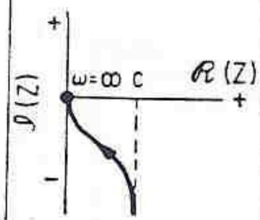
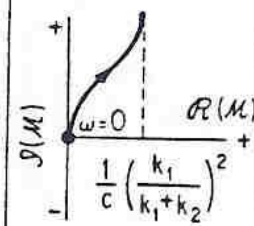
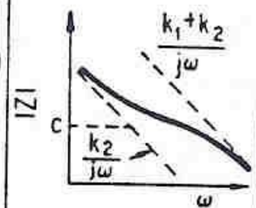
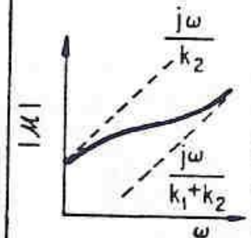
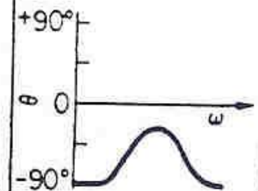
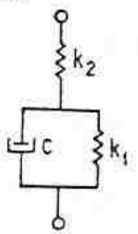
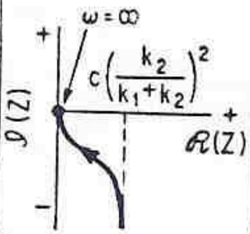
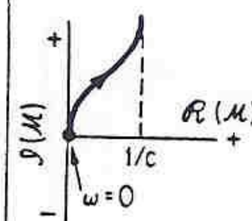
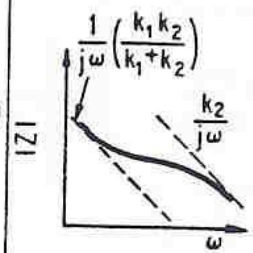
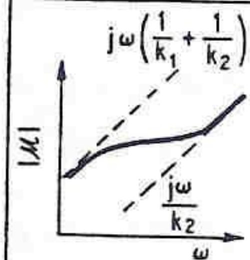
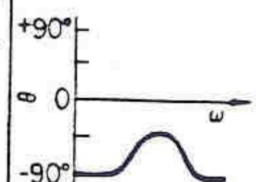
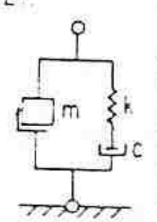
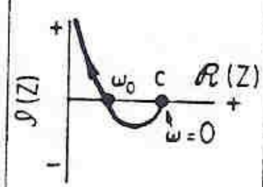
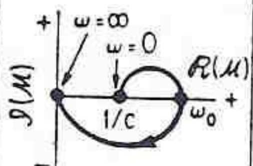
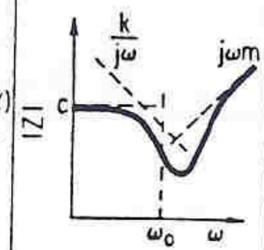
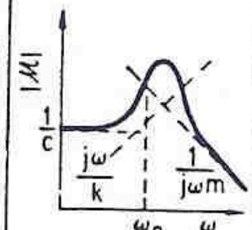
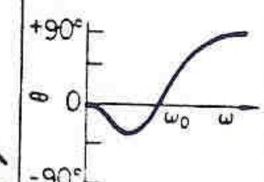
DIAGRAM OF SYSTEM	MATHEMATIC FORMULAS: IMPEDANCE Z - EQ. (10.7) MOBILITY $\mathcal{M}$ - EQ. (10.9)	IMPEDANCE IN THE COMPLEX PLANE	MOBILITY IN THE COMPLEX PLANE	MAGNITUDE OF IMPEDANCE	MAGNITUDE OF MOBILITY	IMPEDANCE ANGLE θ FIG. 10.34
19. 	$Z = \dfrac{\frac{1}{c} - j\left[\frac{k_2}{\omega c^2} + \frac{\omega}{k_1}\left(\frac{k_1+k_2}{k_1}\right)\right]}{(1/c)^2 + (\omega/k_1)^2}$ $\mathcal{M} = \dfrac{\frac{1}{c} + j\left[\frac{k_2}{\omega c^2} + \frac{\omega}{k_1}\left(\frac{k_1+k_2}{k_1}\right)\right]}{\left(\frac{k_1+k_2}{k_1}\right)^2 + \left(\frac{k_2}{\omega c}\right)^2}$					
20. 	$Z = \dfrac{c - j\left[\frac{\omega c^2}{k_2} + \frac{k_1}{\omega}\left(\frac{k_1+k_2}{k_2}\right)\right]}{\left(\frac{k_1+k_2}{k_2}\right)^2 + \left(\frac{\omega c}{k_2}\right)^2}$ $\mathcal{M} = \dfrac{c + j\left[\frac{\omega c^2}{k_2} + \frac{k_1}{\omega}\left(\frac{k_1+k_2}{k_2}\right)\right]}{c^2 + (k_1/\omega)^2}$					
21. 	$Z = \dfrac{\frac{1}{c} + j\omega\left[m\left(\frac{1}{c^2} + \frac{\omega^2}{k^2}\right) - \frac{1}{k}\right]}{(1/c)^2 + (\omega/k)^2}$ $\mathcal{M} = \dfrac{\frac{1}{c} - j\omega\left[m\left(\frac{1}{c^2} + \frac{\omega^2}{k^2}\right) - \frac{1}{k}\right]}{\left(\frac{\omega m}{c}\right)^2 + \left(1 - \frac{\omega^2 m}{k}\right)^2}$ $\omega_0 = \sqrt{\frac{k}{m} - \frac{k^2}{c^2}}$					

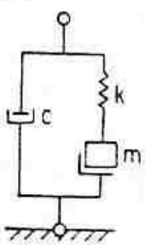
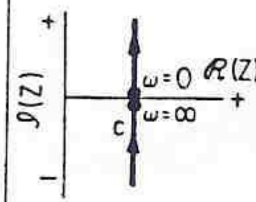
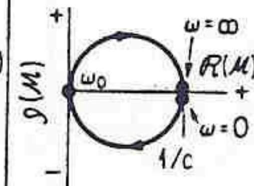
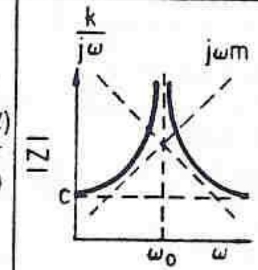
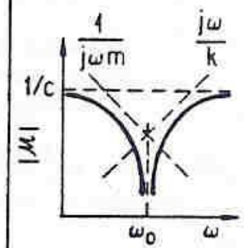
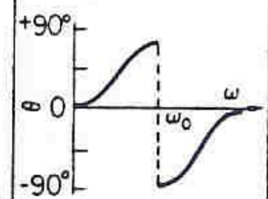
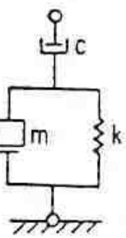
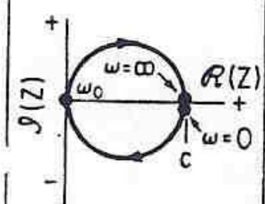
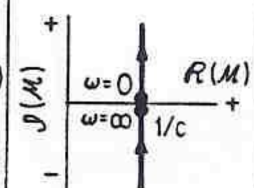
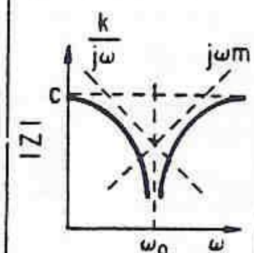
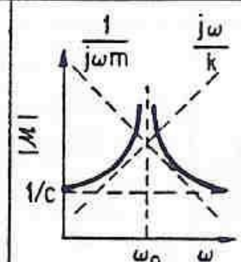
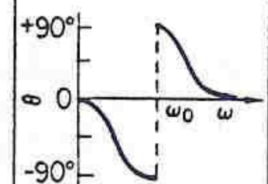
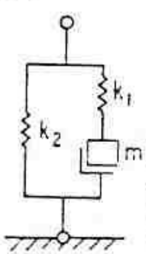
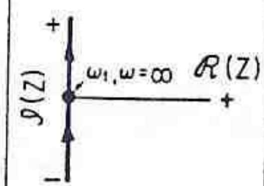
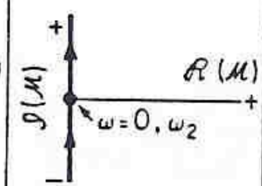
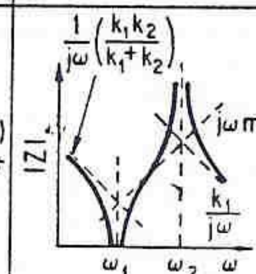
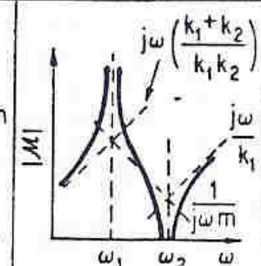
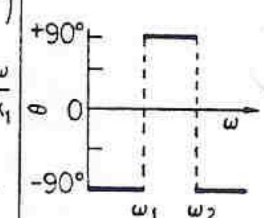
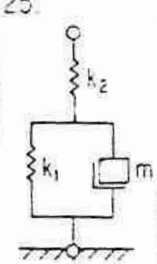
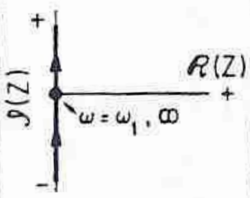
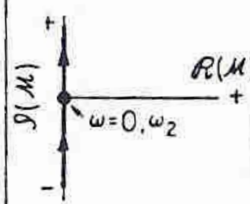
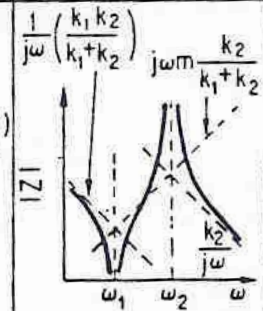
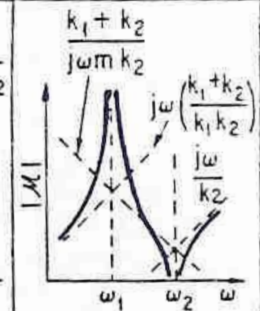
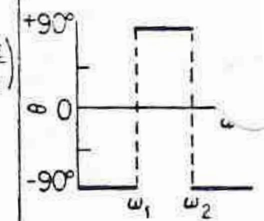
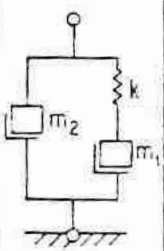
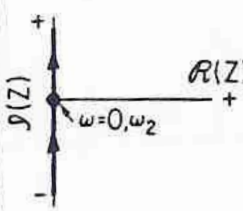
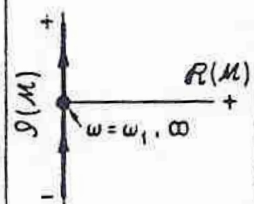
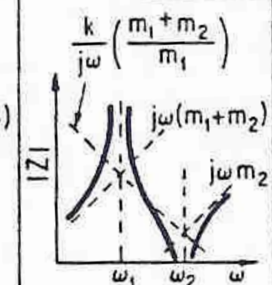
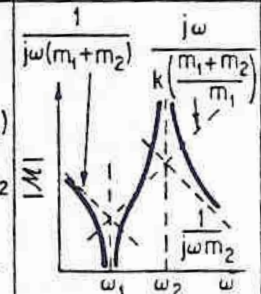
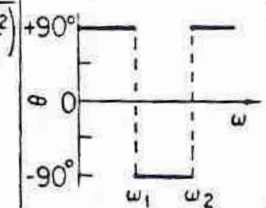
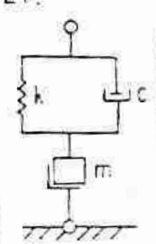
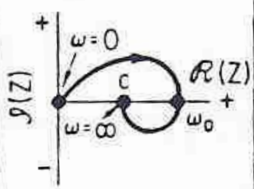
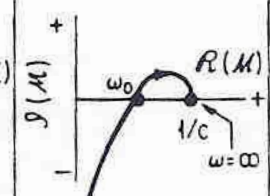
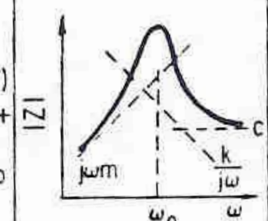
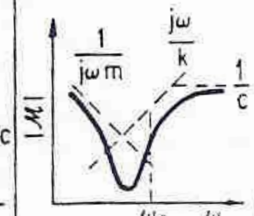
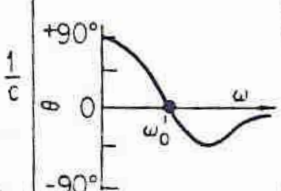
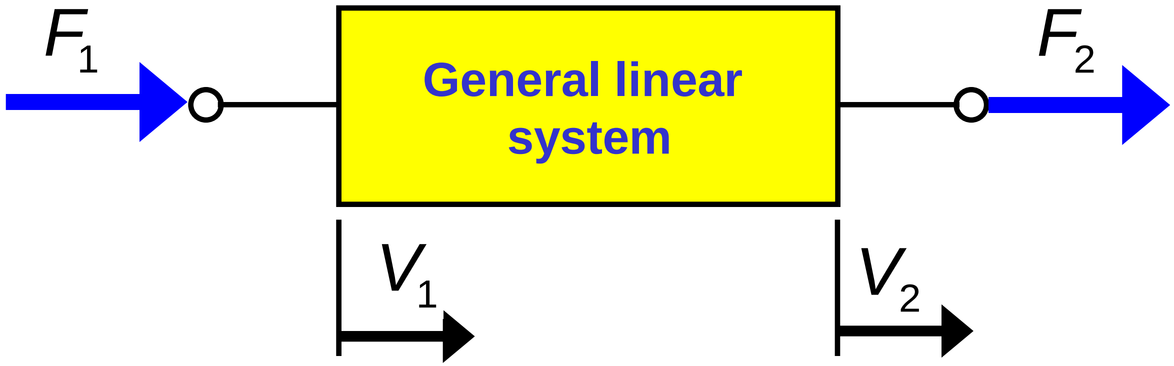
DIAGRAM OF SYSTEM	MATHEMATIC FORMULAS: IMPEDANCE Z - EQ.(10.7) MOBILITY $\mathcal{M}$ - EQ.(10.9)	IMPEDANCE IN THE COMPLEX PLANE	MOBILITY IN THE COMPLEX PLANE	MAGNITUDE OF IMPEDANCE	MAGNITUDE OF MOBILITY	IMPEDANCE ANGLE θ FIG.10.34
22. 	$Z = c + \frac{j\omega m k}{k-\omega^2 m}$ $\mathcal{M} = \frac{c - \frac{j\omega m k}{k-\omega^2 m}}{c^2 + \left(\frac{\omega m k}{k-\omega^2 m}\right)^2}$ $\omega_0 = \sqrt{\frac{k}{m}}$					
23. 	$Z = \frac{\frac{1}{c} - \frac{j\omega}{k-\omega^2 m}}{\left(\frac{1}{c}\right)^2 + \left(\frac{\omega}{k-\omega^2 m}\right)^2}$ $\mathcal{M} = \frac{1}{c} + \frac{j\omega}{k-\omega^2 m}$ $\omega_0 = \sqrt{\frac{k}{m}}$					
24. 	$Z = \frac{-j\left(\frac{k_1+k_2}{k_1} - \frac{k_2}{\omega^2 m}\right)}{\omega/k_1 - 1/\omega m}$ $\mathcal{M} = \frac{+j(\omega/k_1 - 1/\omega m)}{\left(\frac{k_1+k_2}{k_1}\right) - \frac{k_2}{\omega^2 m}}$ $\omega_1 = \sqrt{\frac{1}{m}\left(\frac{k_1 k_2}{k_1+k_2}\right)}$ $\omega_2 = \sqrt{\frac{k_1}{m}}$					

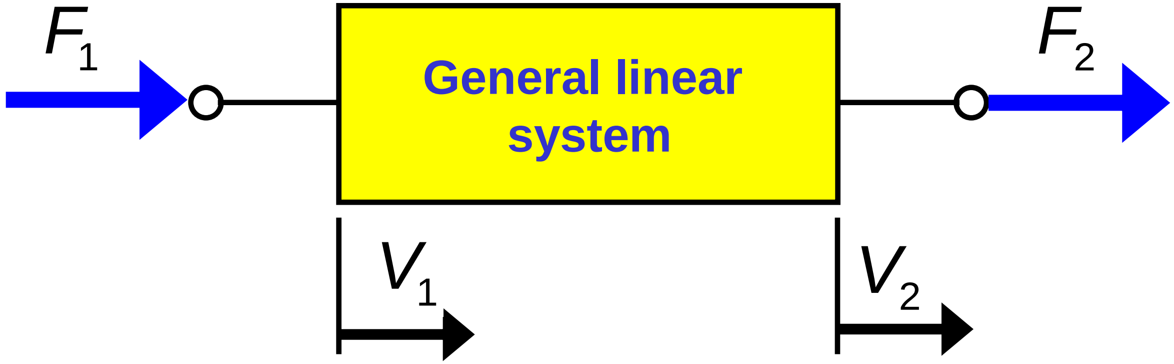
DIAGRAM OF SYSTEM	MATHEMATIC FORMULAS: IMPEDANCE Z - EQ.(10.7) MOBILITY $\mathcal{M}$ - EQ.(10.9)	IMPEDANCE IN THE COMPLEX PLANE	MOBILITY IN THE COMPLEX PLANE	MAGNITUDE OF IMPEDANCE	MAGNITUDE OF MOBILITY	IMPEDANCE ANGLE θ FIG.10.34
25. 	$Z = j\dfrac{\omega m - k_1/\omega}{1 - \dfrac{\omega}{k_2}\left(\omega m - \dfrac{k_1}{\omega}\right)}$ $\mathcal{M} = j\dfrac{\dfrac{\omega}{k_2}\left(\omega m - \dfrac{k_1}{\omega}\right) - 1}{\omega m - k_1/\omega}$ $\omega_1 = \sqrt{\dfrac{k_1}{m}}$ $\omega_2 = \sqrt{\dfrac{k_1 + k_2}{m}}$					
26. 	$Z = +j\left(\omega m_2 + \dfrac{1}{1/\omega m_1 - \omega/k}\right)$ $\mathcal{M} = +j\dfrac{\omega/k - 1/\omega m_1}{\dfrac{m_1+m_2}{m_2} - \dfrac{\omega^2 m_2}{k}}$ $\omega_1 = \sqrt{\dfrac{k}{m_1}}$ $\omega_2 = \sqrt{k\left(\dfrac{1}{m_1} + \dfrac{1}{m_2}\right)}$					
27. 	$Z = \dfrac{\omega m\left\{\omega m c + j\left[c^2 - \dfrac{k}{\omega}\left(\omega m - \dfrac{k}{\omega}\right)\right]\right\}}{c^2 + (\omega m - k/\omega)^2}$ $\mathcal{M} = \dfrac{c + \dfrac{1}{j\omega m}\left[c^2 - \dfrac{k}{\omega}\left(\omega m - \dfrac{k}{\omega}\right)\right]}{c^2 + (k/\omega)^2}$ $\omega_0 = \sqrt{\dfrac{k}{m - c^2/k}}$					

Coupling Together Complex Arbitrary Systems



- An arbitrary system having only two inputs can be represented using the sign convention shown in the figure above (F_1 and F_2 both act on the element).
- When the system is not a simple mass, spring or damper it is necessary to assign both ***point*** and ***transfer mobilities*** to a system to define completely the inter-relationship between both inputs.

Mobility Method



- The equations describing the system are given by

$$V_1 = Y_{11}F_1 + Y_{12}F_2$$
$$V_2 = Y_{21}F_1 + Y_{22}F_2$$

which can be written in matrix form as

$$\begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}}_{\text{Mobility matrix}} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

or

$$\mathbf{v} = \mathbf{Y}\mathbf{f}$$

Mobility Method

$$\begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Y_{11} and Y_{22} are ***point*** mobilities, which relate the velocity at the point of excitation to the force applied. They are defined as

$$Y_{11} = \left.\frac{V_1}{F_1}\right|_{F_2=0} \qquad\qquad Y_{22} = \left.\frac{V_2}{F_2}\right|_{F_1=0}$$

Force applied at point 1

Point 2 Is free

Force applied at point 2

Point 1 Is free

Y_{12} and Y_{21} are ***transfer*** mobilities, which relate the velocity at the point of some remote point to the force applied. They are defined as

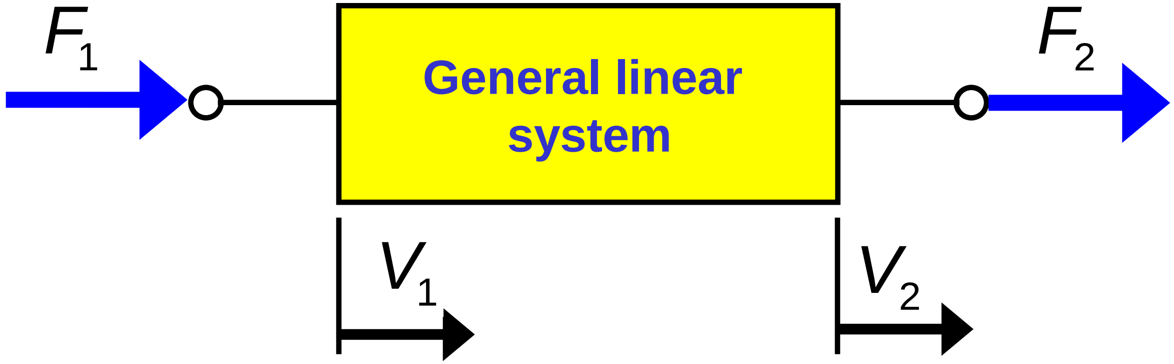
$$Y_{12} = \left.\frac{V_1}{F_2}\right|_{F_1=0} \qquad\qquad Y_{21} = \left.\frac{V_2}{F_1}\right|_{F_2=0}$$

Mobility Method

Notes

- For a linear system $Y_{12} = Y_{21}$ because of ***reciprocity***
- If the system is ***symmetric*** then $Y_{11} = Y_{22}$
- These mobilities are found by allowing one input to be free
- The mobility formulation is most useful in experimental work as the point and transfer mobilities are easily measured
- $\mathrm{Re}\{Y_{11}\}$ and $\mathrm{Re}\{Y_{22}\}$ must be positive

Impedance Method



- The equations describing the system are given by

$$F_1 = Z_{11}V_1 + Z_{12}V_2$$
$$F_2 = Z_{21}V_1 + Z_{22}V_2$$

which can be written in matrix form as

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix}$$

Impedance matrix

or $$\mathbf{f} = \mathbf{Z}\mathbf{v}$$

Impedance Method

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix}$$

Z_{11} and Z_{22} are ***point*** impedances, which relate the velocity at the point of excitation to the force applied. They are defined as

$$Z_{11} = \left.\frac{F_1}{V_1}\right|_{V_2=0} \qquad Z_{22} = \left.\frac{F_2}{V_2}\right|_{V_1=0}$$

Force applied at point 1

Point 2 Is fixed

Force applied at point 2

Point 1 Is fixed

Z_{12} and Z_{21} are ***transfer*** impedances, which relate the velocity at the point of some remote point to the force applied. They are defined as

$$Z_{12} = \left.\frac{F_1}{V_2}\right|_{V_1=0} \qquad Z_{21} = \left.\frac{F_2}{V_1}\right|_{V_2=0}$$

Impedance Method

Notes

- For a linear system $Z_{12} = Z_{21}$ because of ***reciprocity***
- If the system is ***symmetric*** then $Z_{11} = Z_{22}$
- These impedances are found by fixing all points except one
- The impedance formulation is most useful for theoretical formulation and when experimentally working on light structures when blocking is possible
- $\text{Re}\{Z_{11}\}$ and $\text{Re}\{Z_{22}\}$ must be positive

Impedance and Mobility matrices for simple elements

Spring

$$\mathbf{Z}_k = \frac{k}{j\omega}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\mathbf{Y}_k$ is not defined as the element is massless, and the mobilities are infinite if one input is free

Damper

$$\mathbf{Z}_c = c\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

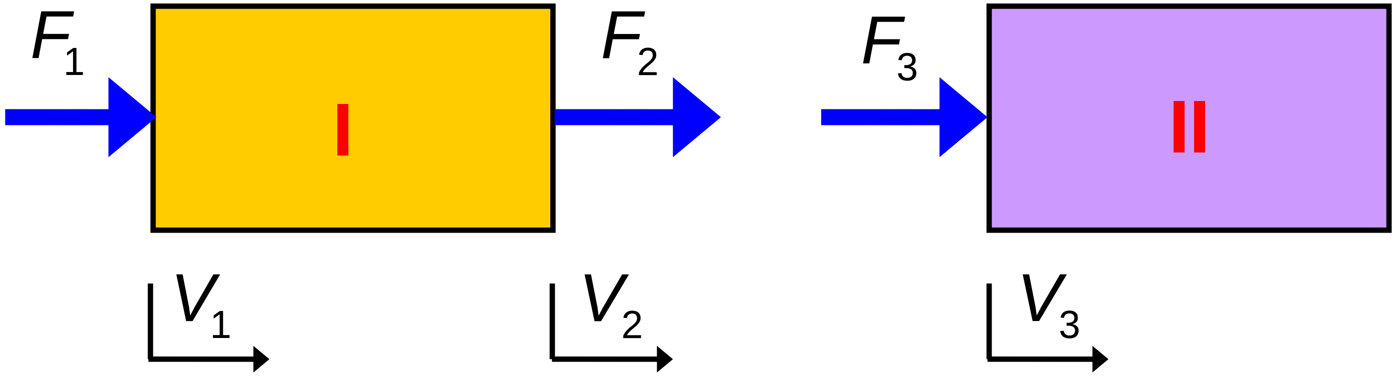
$\mathbf{Y}_c$ is not defined as the element is massless, and the mobilities are infinite if one input is free

Mass

$$\mathbf{Y}_m = \frac{-j}{m\omega}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$\mathbf{Z}_m$ is not defined as a blocked output would prevent motion

Coupling together complex arbitrary systems



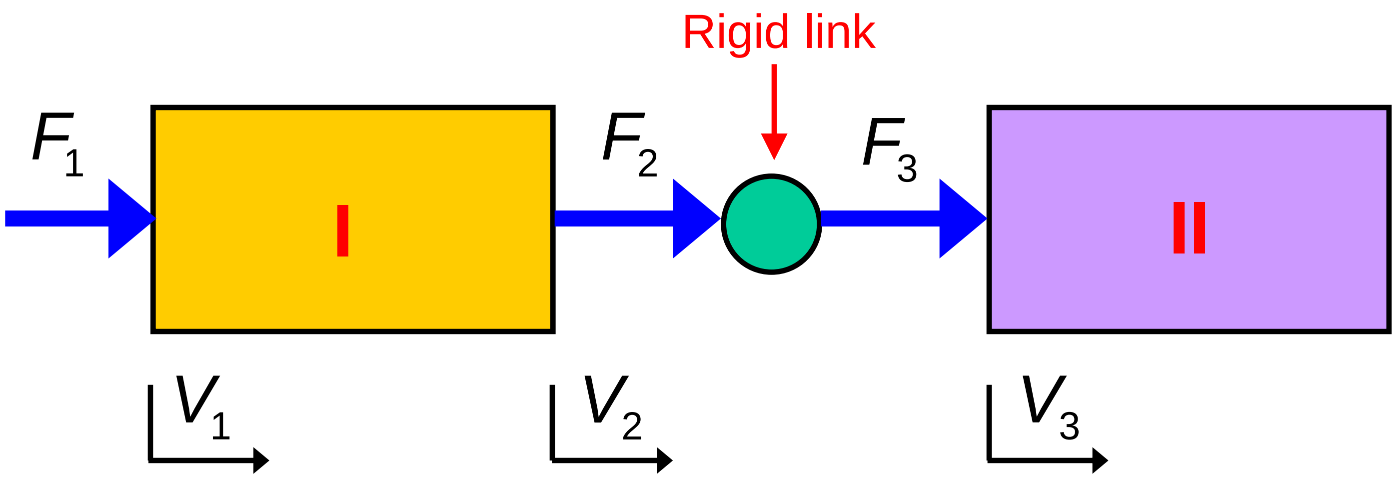
System I

$$V_1 = Y_{11}F_1 + Y_{12}F_2$$
$$V_2 = Y_{21}F_1 + Y_{22}F_2$$

System II

$$V_3 = Y_{33}F_3$$

Coupling together complex arbitrary systems - series connection



When rigidly connected $F_3 = -F_2$ (equilibrium of forces)

$V_3 = V_2$ (continuity of motion)

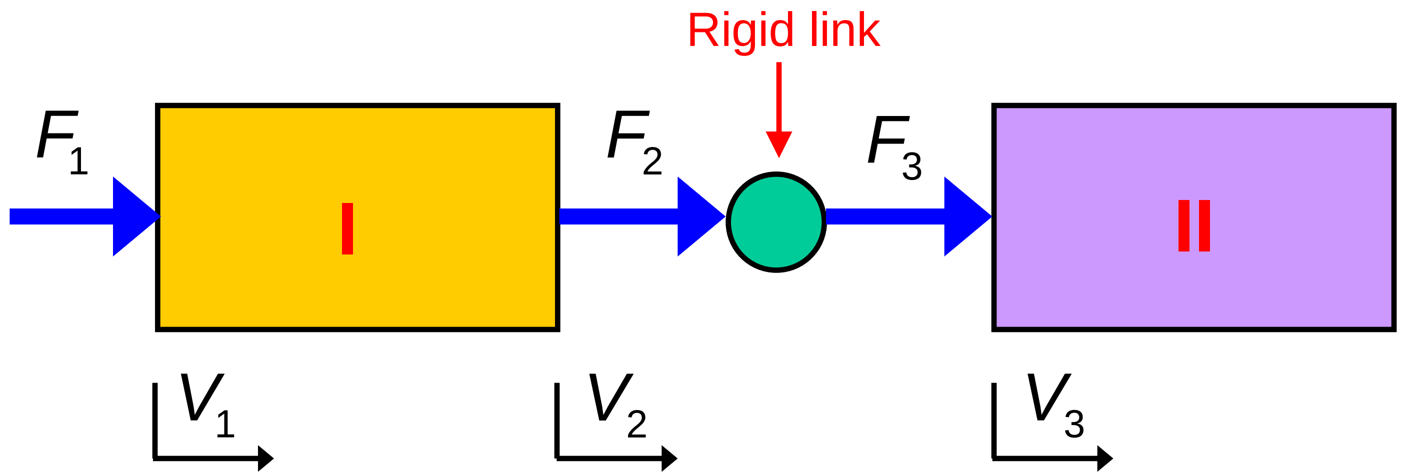
System I

$$V_1 = Y_{11}F_1 + Y_{12}F_2$$
$$V_2 = Y_{21}F_1 + Y_{22}F_2$$

System II

$$V_3 = Y_{33}F_3$$

Coupling together complex arbitrary systems - series connection



Combining system equations gives the point and transfer mobilities of the coupled system

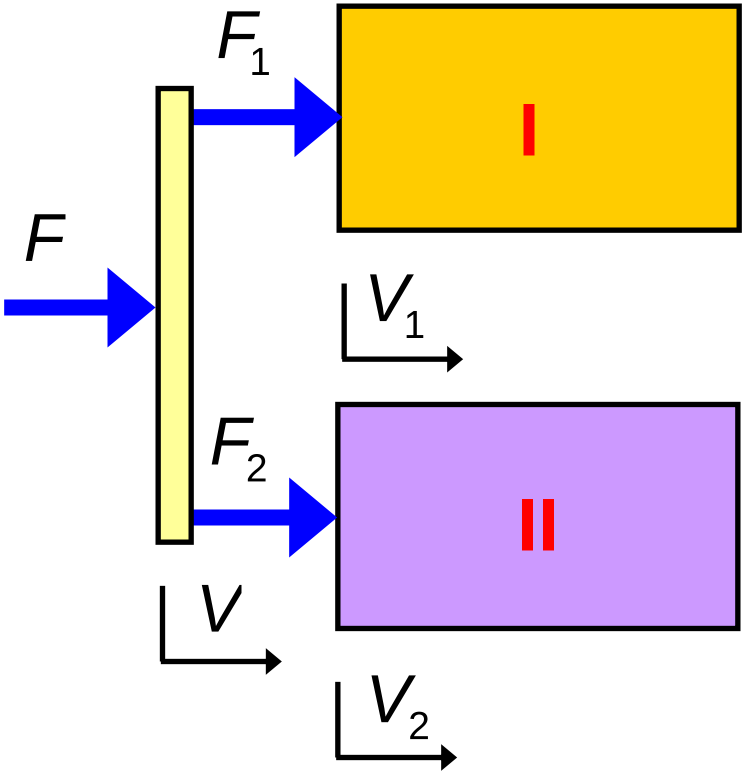
Point $$\frac{V_1}{F_1} = Y_{11} - \frac{(Y_{21})^2}{Y_{22} + Y_{33}}$$

Transfer $$\frac{V_2}{F_1} = \frac{Y_{12} Y_{33}}{Y_{22} + Y_{33}}$$

The natural frequencies of the **coupled** system occur when $\mathrm{Im}\{Y_{22} + Y_{33}\} = 0$

The imaginary components embody the reactive elements which can equal zero

Coupling together complex arbitrary systems - parallel coupled system



For the uncoupled systems

$$F_1 = Z_{11} V_1$$

$$F_2 = Z_{22} V_2$$

When the systems are joined by a rigid link

$$F = F_1 + F_2$$

$$V = V_1 = V_2$$

so $\dfrac{F}{V} = Z_{11} + Z_{22}$

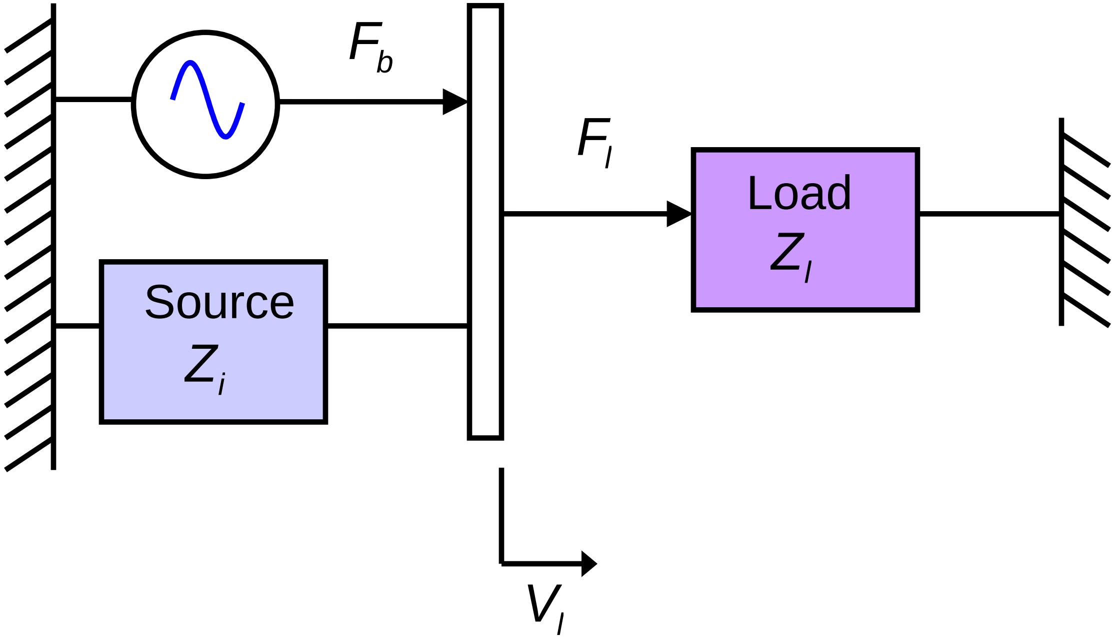
The point mobility of the coupled system is given by

$$\frac{V}{F} = \frac{1}{Z_{11} + Z_{22}} = \frac{1}{1/Y_{11} + 1/Y_{22}}$$

Vibration source characterisation

Thévenin equivalent system

- A vibration source connected to a load can be represented by a blocked force F_b in parallel with an internal impedance Z_i connected to a load impedance Z_l.



$$F_l = F_b \frac{1}{1 + \frac{Z_i}{Z_l}}$$

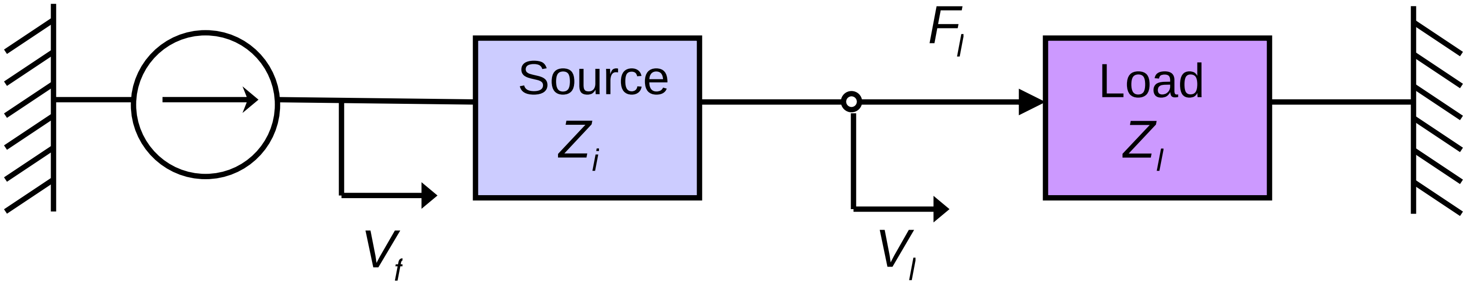
$$V_l = \frac{F_b}{Z_i + Z_l}$$

Blocked force is the force generated by the source when it is connected to rigid load

Vibration source characterisation

Norton equivalent system

- A vibration source connected to a load can be represented by the free velocity of the source V_f in series with an internal impedance Z_i connected to a load impedance Z_l.



$$V_l = V_f \frac{1}{1+\frac{Z_l}{Z_i}}$$

$$F_l = V_f \frac{Z_l}{1+\frac{Z_l}{Z_i}}$$

Free velocity is the velocity of the source when the load is disconnected

Vibration source characterisation

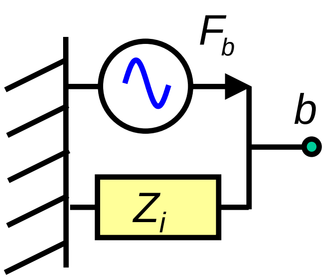
- Relationship between blocked force, free velocity and internal impedance of the source

Thévenin equivalent system

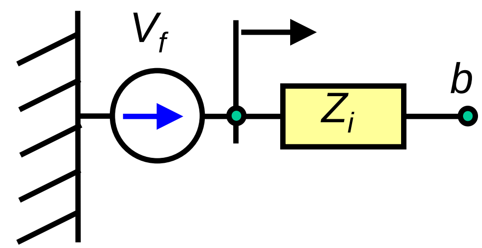
Norton equivalent system

$$V_l = \frac{F_b}{Z_i + Z_l}$$

$$V_l = \frac{Z_i}{Z_i + Z_l} V_f$$



$$F_b = Z_i \, V_f$$

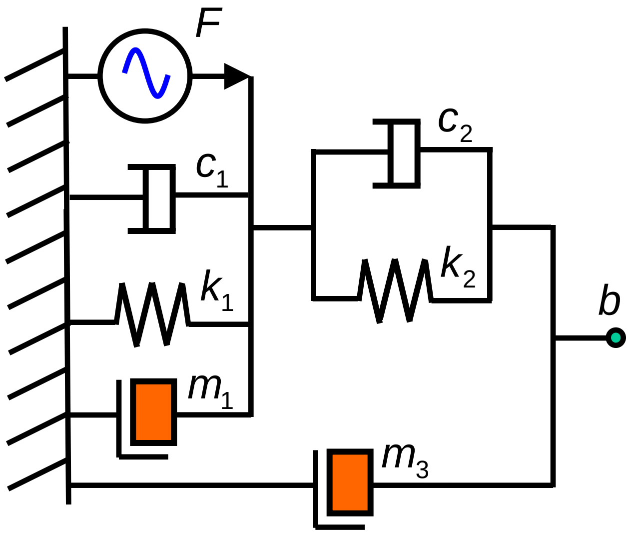


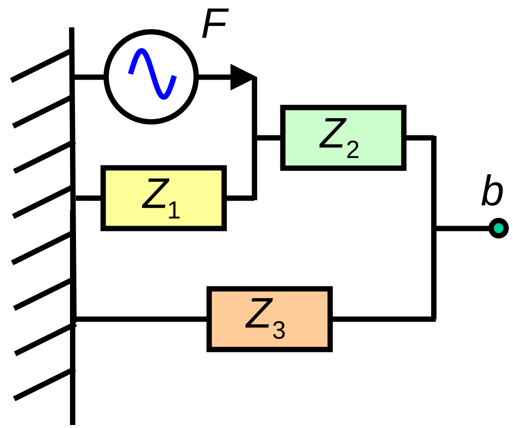
$$F_l = \frac{Z_l}{Z_i + Z_l} F_b$$

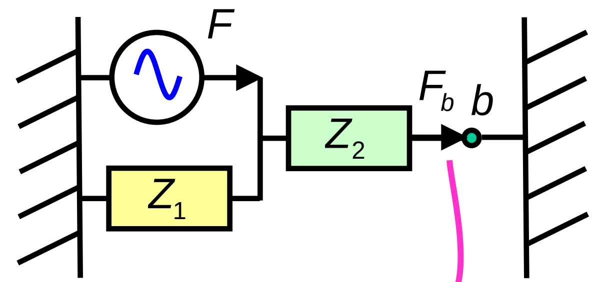
$$F_l = \frac{Z_i Z_l}{Z_i + Z_l} V_f$$

- Similar equations involving mobilities.

Reduction of a system to Thévenin and Norton equivalent systems – example







$$Z_1 = Z_{m1} + Z_{k1} + Z_{c1}$$

$$Z_2 = Z_{k2} + Z_{c2}$$

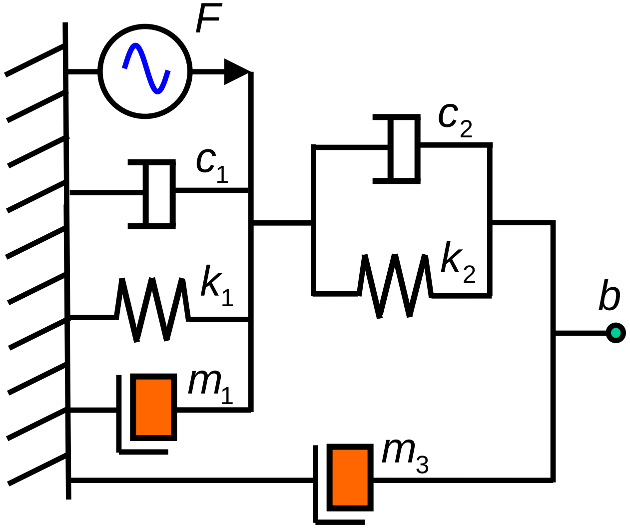
$$Z_3 = Z_{m3}$$

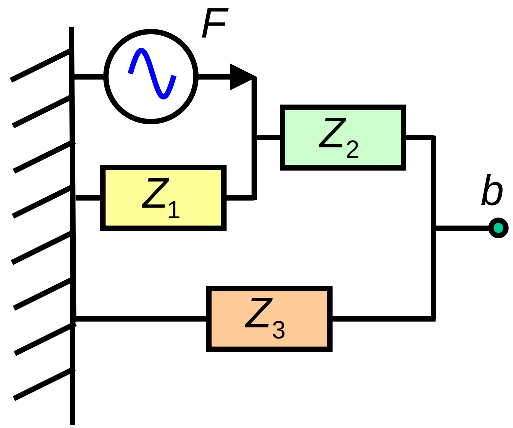
Blocked Force

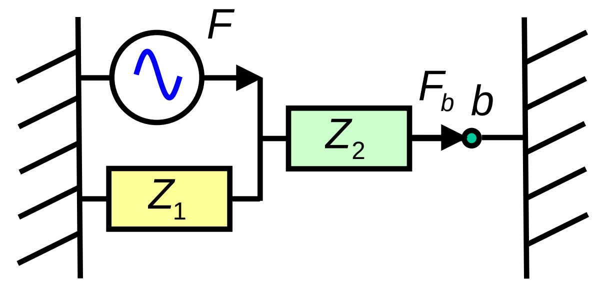
$$\boxed{F_b = \frac{Z_2}{Z_1 + Z_2} F}$$

Z_3 is not included because its ends are attached to two rigid walls

Reduction of a system to Thévenin and Norton equivalent systems – example

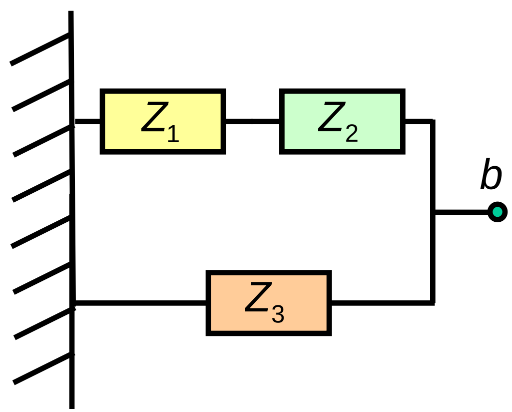






Free velocity

$$\boxed{V_f = \frac{F_b}{Z_s}}$$

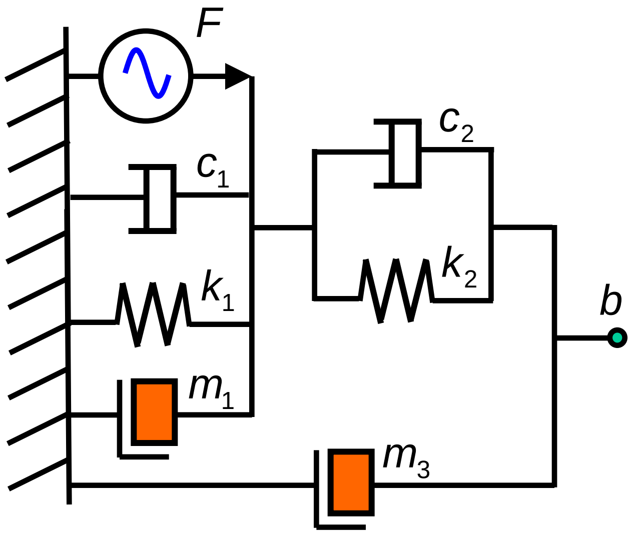


Source impedance
at connection point

Internal impedance

$$\boxed{Z_s = Z_3 + \frac{1}{1/Z_1 + 1/Z_2}}$$

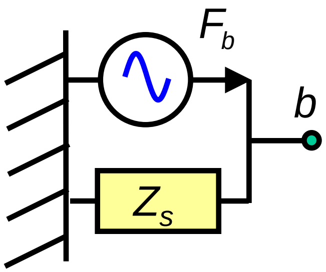
Reduction of a system to Thévenin and Norton equivalent systems – example

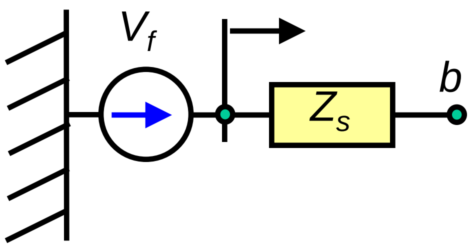


$$Z_1 = Z_{m1} + Z_{k1} + Z_{c1}$$

$$Z_2 = Z_{k2} + Z_{c2}$$

$$Z_3 = Z_{m3}$$





Blocked Force

$$F_b = \frac{Z_2}{Z_1 + Z_2} F$$

Free velocity

$$V_f = \frac{F_b}{Z_s}$$

Internal impedance

$$Z_s = Z_3 + \frac{1}{1/Z_1 + 1/Z_2}$$

Summary

- Introduction to mobility and impedance approach
- Lumped parameter systems
- Arbitrary systems
- Coupling of systems
- Source characterisation

References

- E.L. Hixson, 1997. Shock and Vibration Handbook (Chapter 10), edited by C.M. Harris, Third Edition, McGraw Hill. Mechanical Impedance.
- R.E.D. Bishop and D.C. Johnson, 1960. The Mechanics of Vibration, Cambridge University Press.
- L. Cremer, M. Heckl and E.E. Ungar, 1990, Structure-Borne Sound, Springer-Verlag, Berlin-Heidelberg-New York.
- L.L. Beranek and I.L. Ver, 1992. Noise and Vibration Control Engineering, John Wiley and Sons.
- F.J. Fahy and J.G. Walker, 2004, Advanced Applications in Acoustics, Noise and Vibration, Spon Press (chapter 9 Mobility and impedance methods in structural dynamics by P. Gardonio and M.J. Brennan).